On Negative Order KdV Equations

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Abstract.

In this paper, we study negative order KdV (NKdV) equations and give their Hamiltonian structures, Lax pairs, infinitely many conservation laws, and explicit multi-soliton and multi-kink wave solutions thorough bilinear Bäcklund transformations. The NKdV equations studied in the paper are differential and can be derived from the first member in the negative order KdV hierarchy. The NKdV equations are not only gauge-equivalent to the Camassa-Holm equation through some hodograph transformations, but also closely related to the Ermakov-Pinney systems and the Kupershmidt deformation. The bi-Hamiltonian structures and a Darboux transformation of the NKdV equations are constructed with the aid of trace identity and their Lax pairs, respectively. The 1- and 2- kink wave and soliton solutions are given in an explicit formula through the Darboux transformation. The 1-kink wave solution is expressed in the form of $tanh$ while the 1-bell soliton is in the form of $sech$, and both forms are very standard. The collisions of 2-kink-wave and 2-bell-soliton solutions, are analyzed in details, and this singular interaction is a big difference from the regular KdV equation. Multi-dimensional binary Bell polynomials are employed to find bilinear formulation and Bäcklund transformations, which produce N-soliton solutions. A direct and unifying scheme is proposed for explicitly building up quasi-periodic wave solutions of the NKdV equations. Furthermore, the relations between quasi-periodic wave solutions and soliton solutions are clearly described. Finally, we show the quasi-periodic wave solution convergent to the soliton solution under some limit conditions.

Keywords: Negative order KdV equations, bilinear Bäcklund transformation, Darboux transformation, kink wave solution, soliton solution, quasi-periodic solution.

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1. Introduction

The Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0$$

was proposed by Korteweg and de Vries in fluid dynamics [38], starting from the observation and subsequent experiments by Russell [76]. There are many excellent sources for the highly interesting background and historical development of the KdV equation, which brings it to the forefront of modern mathematical physics. In 1967, Gardner, Greener, Kruskal and Miura found the inverse scattering transformation method to solve the Cauchy problem of the KdV equation with sufficiently decaying initial data [18]. Soon thereafter, Lax explained the magical isospectral property of the time dependent family of Schrodinger operators which is now called the Lax pair, and introduced the KdV hierarchy through a recursive procedure [43]. In the same year a sequence of infinitely many polynomial conservation laws were obtained with the help of Miura's transformation [53, 54].

There are tools to view the KdV equation as a completely integrable system by Gardner, and Zakharov and Faddeev [19, 81]. The bilinear derivative method was developed by Hirota to find N-soliton solutions of the KdV equation [28]. The KdV hierarchy was constructed by Lax [42] through a recursive approach, and further studied by Gel'fand and Dikii [20]. On the base of the inverse spectral theory and algebro-geometric methods, the inverse scattering method was extended to periodic initial data by by Novikov, Dubrovin, Lax, Its, Matveev et al [11, 36, 49, 57]. For more recent reviews on the KdV equation one may refer to literature [1, 3, 4, 5, 8, 25, 46, 59, 77, 78].

All the work done in the above mentioned publications dealt with the positive order KdV hierarchy, which includes the KdV equation as a special member. However, there was only little work on the NKdV hierarchy. Verosky [80] studied symmetries and negative powers of recursion operator and gave the following

negative order KdV equation (called the NKdV equation thereafter)

$$\begin{aligned} &v_t = w_x, \\ &w_{xxx} + 4vw_x + 2v_x w = 0. \end{aligned} \tag{1.1}$$

and Lou [47] presented additional symmetries based on the invertible recursion operator of the KdV system and particularly provided the following NKdV equation (called the NKdV-1 equation thereafter)

$$v_t = 2uu_x, \quad u_{xx} + vu = 0, \Longleftrightarrow \left(\frac{u_{xx}}{u}\right)_t + 2uu_x = 0, \tag{1.2}$$

which can be reduced from the NKdV equation (1.1) under the following transformation

$$w = u^2, \quad v = -\frac{u_{xx}}{u}. \tag{1.3}$$

Moreover, the second part of NKdV-1 equation (1.2) is a linear Schrödinger equation or Hill equation

$$u_{xx} + vu = 0.$$

Fuchssteiner [17] pointed out the gauge-equivalent relation between the NKdV equation (1.1) and the Camassa-Holm (CH) equation [6]

$$m_t + m_x u + 2mu_x = 0, \quad m = u - u_{xx}$$

through some hodograph transformation, and later on Hone proposed the associate CH equation, which is actually equivalent to the NKdV equation (1.1), and gave soliton solutions through the KdV system [32]. Zhou [85] generalized the Kupershmidt deformation and proposed a kind of the mixed KdV hierarchy, which contains the NKdV equation (1.1) as a special case.

Very recently, Qiao and Li [61] gave a unifying formulation of the Lax representations for both negative and positive order KdV hierarchies, and furthermore studied all possible traveling wave solutions, including soliton, kink wave, and periodic wave solutions, of the integrable NKdV-1 equation (1.2) with the following Lax pair

$$\begin{aligned} &L\psi \equiv \psi_{xx} + v\psi = \lambda\psi, \\ &\psi_t = \tfrac{1}{2}u^2\lambda^{-1}\psi_x - \tfrac{1}{2}uu_x\lambda^{-1}\psi. \end{aligned} \tag{1.4}$$

The most interesting [61] is: the NKdV-1 equation has both soliton and kink solutions, which is the first integrable example, within our knowledge, having such a property in soliton theory.

Studying negative order integrable hierarchies plays an important role in the theory of peaked soliton (peakon) and cusp soliton (cuspon). For instance, the well-known CH peakon equation is actually produced through its negative order hierarchy while its positive order hierarchy includes the remarkable Harry-Dym type equation [62]. The Degasperis-Procesi (DP) peakon equation [9] can also be generated through its negative order hierarchy [64]. Both the CH equation and the DP equation are typical integrable peakon and cupson systems with nonlinear quadratic terms [6, 10, 48, 62, 83]. Recently, some nonlinear cubic integrable equations have also been found to have peakon and cupson solutions [33, 58, 63, 65].

In this paper, we study the NKdV hierarchy, in particular, focus on the NKdV equation (1.1) and the NKdV-1 equation (1.2). Actually, as per [47, 66], the NKdV equation (1.1) can embrace other possible differential-integro forms according to the kernel of operator $K = \frac{1}{4}\partial_x^3 + \frac{1}{2}(v\partial_x + \partial_x v)$. Here we just list the NKdV-1 equation (1.2) as it is differential and also equivalent to a nonlinear quartic integrable system:

$$uu_{xxt} - u_{xx}u_t - 2u^3u_x = 0.$$

The purpose of this paper is to investigate integrable properties, N-soliton and N-kink solutions of the NKdV equation (1.1) and NKdV-1 equation (1.2). In section 2, the trace identity technique is employed to construct the bi-Hamiltonian structures of the NKdV hierarchy. In section 3, we show that the NKdV equation (1.1) is related to the Kupershmidt deformation and the Ermakov-Pinney systems, and is also able to reduced to the NKdV-1 equation (1.2) under a transformation. In section 4, a Darboux transformation of the NKdV equation (1.1) is provided with the help of its Lax pairs. In section 5, as a direct application of the Darboux transformation, the kink-wave and bell soliton solutions are explicitly given, and the collision of two soliton solutions is analyzed in detail through

two-solitons. In section 6, an extra auxiliary variable is introduced to bilinearize the NKdV equation (1.1) through binary Bell polynomials. In section 7, the bilinear Bäcklund transformations are obtained and Lax pairs are also recovered. In section 8, we will give a kind of Darboux covariant Lax pair, and in section 9, infinitely many conservation laws of the NKdV equation (1.1) are presented through its Lax equation and a generalized Miura transformation. All conserved densities and fluxes are recursively given in an explicit formula. In sections 10, a direct and unifying scheme is proposed for building up quasi-periodic wave solutions of the NKdV equation (1.1) in an explicit formula. Furthermore, the relations between quasi-periodic wave solutions and soliton solutions are clearly described. Finally, we show the quasi-periodic wave solution convergent to the soliton solution under some limit conditions.

2. Hamiltonian structures of the NKdV hierarchy

To find the Hamiltonian structures of the NKdV hierarchy, let us re-derive the NKdV hierarchy in matrix form.

2.1. The NKdV hierarchy

Consider the Schrödinger-KdV spectral problem

$$\psi_{xx} + v\psi = \lambda\psi, \tag{2.1}$$

where λ is an eigenvalue, ψ is the eigenfunction corresponding to the eigenvalue λ, and v is a potential function.

Let $\varphi_1 = \psi, \quad \varphi_2 = \psi_x$, then the spectral problem (2.1) becomes

$$\varphi_x = U\varphi = \begin{pmatrix} 0 & 1 \\ \lambda - v & 0 \end{pmatrix} \varphi, \tag{2.2}$$

where $\varphi = (\varphi_1, \varphi_2)^T$ is a two-dimensional vector of eigenfunctions.

The Gateaux derivative of spectral operator U in direction ξ at point v is

$$U'[\xi] = \frac{d}{d\varepsilon} U(v + \varepsilon\xi)|_{\varepsilon=0} = \begin{pmatrix} 0 & 0 \\ -\xi & 0 \end{pmatrix}, \tag{2.3}$$

which is injective and linear with respect to the variable ξ.

The Lenard recursive sequence $\{G_m\}$ of the spectral problem (2.1) is defined by

$$\begin{aligned} &G_{-1} \in KerK = \{G|KG = 0\}, \quad G_0 \in KerJ = \{G|JG = 0\} \\ &KG_{m-1} = JG_m, \quad m = 0, -1, -2 \cdots , \end{aligned} \tag{2.4}$$

which directly produces the NKdV hierarchy:

$$v_t = KG_{m-1} = JG_m, \quad m = -1, -2 \cdots \tag{2.5}$$

where

$$K = \frac{1}{4}\partial_x^3 + \frac{1}{2}(v\partial_x + \partial_x v), \qquad J = \partial_x, \tag{2.6}$$

and K is exactly a recursion operator of the well-known KdV hierarchy

$$v_t = K^n v_x, \quad n = 0, 1, 2, \cdots .$$

The first equation ($m = 0$) in the NKdV hierarchy (2.5) is trivial equation

$$v_t = JG_0 = 0, \quad JG_0 = KG_{-1} = 0.$$

The second equation ($m = -1$) in the NKdV hierarchy (2.5) takes

$$v_t = G_{-1,x}, \qquad KG_{-1} = 0,$$

which is exactly the NKdV equation (1.1) by replacing $G_{-1} = w$.

In a similar way to the paper [61], we construct zero curvature representation for NKdV hierarchy.

Proposition 1. Let U be the spectral matrix defined in (2.2), then for an arbitrarily smooth function $G \in C^\infty(\mathbb{R})$, the following operator equation

$$V_x - [U, V] = U'[KG] - \lambda U'[JG] \tag{2.7}$$

admits a matrix solution

$$V = V(G) = \begin{pmatrix} -\frac{1}{4}G_x & \frac{1}{2}G \\ -\frac{1}{4}G_{xx} - \frac{1}{2}vG + \frac{1}{2}\lambda G & \frac{1}{4}G_x \end{pmatrix} \lambda^{-1},$$

which is a linear function with respect to G, and Gateaux derivative is defined by (2.3).

Theorem 1. Suppose that $\{G_j, \quad j = -1, -2, \cdots\}$ is the first Lenard sequence defined by (2.4), and $V_j = V(G_j)$ is a corresponding solution to the

operator equation (2.7) for $G = G_j$. With V_j being its coefficients, a mth matrix polynomial in λ is constructed as follows

$$W_m = \sum_{j=1}^{m} V_j \lambda^{-m+j}.$$

Then we conclude that the NKdV hierarchy (2.5) admits zero curvature representation

$$U_t - W_{m,x} + [U, W_m] = 0,$$

which is equivalent to

$$\begin{aligned} &\varphi_x = U\varphi = \begin{pmatrix} 0 & 1 \\ \lambda - v & 0 \end{pmatrix} \varphi, \\ &\varphi_t = W_m \varphi = \sum_{j=1}^{m} \begin{pmatrix} -\frac{1}{4}G_{j,x} & \frac{1}{2}G_j \\ -\frac{1}{4}G_{j,xx} - \frac{1}{2}vG_j + \frac{1}{2}\lambda G_j & \frac{1}{4}G_{j,x} \end{pmatrix} \lambda^{-m+j-1}\varphi. \end{aligned} \tag{2.8}$$

This theorem actually provides an unified formula of the Lax pairs for the whole NKdV hierarchy (2.5).

According to theorem 1, the NKdV equation (1.1) admits Lax pair with parameter λ

$$L\psi \equiv \psi_{xx} + v\psi = \lambda\psi,$$

$$\psi_t = \tfrac{1}{2}w\lambda^{-1}\psi_x - \tfrac{1}{4}w_x\lambda^{-1}\psi,$$

or equivalently,

$$\begin{aligned} &L\psi = (\partial_x^2 + v)\psi = \lambda\psi, \\ &M\psi = (4\partial_x^2\partial_t + 4v\partial_t + 2w\partial_x + 3w_x)\psi = 0. \end{aligned} \tag{2.9}$$

The NKdV equation (1.1) also possesses Lax pair without parameter

$$\begin{aligned} &L\psi = (\partial_x^2 + v)\psi = 0, \\ &M\psi = (4\partial_x^2\partial_t + 4v\partial_t + 2w\partial_x + 3w_x)\psi = 0. \end{aligned} \tag{2.10}$$

Especially, taking the constraint $v = -u_{xx}/u$ and $w = u^2 \in KerK$, we then further get the NKdV equation (1.2) and its Lax parir (1.4).

2.2. Hamiltonian structures

Proposition 2. [77] For the spectral problem (2.2), assume that V is a solution to the following stationary zero curvature equation with the given homogeneous rank

$$V_x = [U, V]. \tag{2.11}$$

Then there exists a constant β, such that

$$\frac{\delta}{\delta v}\left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle = \left(\lambda^{-\beta}\frac{\partial}{\partial \lambda}\lambda^{\beta}\right)\left\langle V, \frac{\partial U}{\partial v}\right\rangle, \tag{2.12}$$

holds, where $\langle \cdot, \cdot \rangle$ stands for the trace of the product of two matrices.

Let $\{G_m,\ m = -1, -2\cdots\}$ be the negative order Lenard sequence recursively given through (2.4) and

$$G_\lambda = \sum_{m=-\infty}^{-1} G_m \lambda^{-m}, \tag{2.13}$$

be a series with respect to λ. Assume that $V_\lambda = V(G_\lambda)$ is the matrix solution for the operator equation (2.9) corresponding to $G = G_\lambda$. So, V_λ can be written as

$$V_\lambda = \sum_{m=-\infty}^{-1} V_m \lambda^{-m}.$$

Then, we have the following proposition.

Proposition 3. V_λ satisfies the following Lax form

$$V_{\lambda,x} = [U, V_\lambda].$$

Proof. By (2.4), we have

$$\begin{aligned}(K - \lambda J)G_\lambda &= \sum_{m=-\infty}^{-1} KG_m\lambda^{-m} - \sum_{m=-\infty}^{-1} JG_m\lambda^{-m+1} \\ &= KG_{-1}\lambda^{-1} + \sum_{m=-\infty}^{-1}(KG_{m-1} - JG_m)\lambda^{-m} = 0.\end{aligned}$$

Therefore, Proposition 1 implies

$$V_{\lambda,x} - [U, V_\lambda] = U'[KG_\lambda] - \lambda U'[JG_\lambda] = U'[KG_\lambda - \lambda JG_\lambda] = 0.$$

□

Next, we discuss the Hamiltonian structures of the hierarchy (2.5). It is crucial to find infinitely many conserved densities.

Theorem 2.

(1) The hierarchy (2.5) possesses the bi-Hamiltonian structures

$$v_t = K\frac{\delta H_{m-1}}{\delta v} = J\frac{\delta H_m}{\delta v}, \quad m = -1, -2\cdots, \tag{2.14}$$

where the Hamiltonian functions H_m are implicitly given through the following formulas

$$H_{-1} = G_{-1} \in KerK, \quad H_m = \frac{G_m}{m}, \quad m = -1, -2\cdots. \tag{2.15}$$

(2) The hierarchy (2.5) is integrable in the Liouville sense.

(3) The Hamiltonian functions $\{H_m\}$ are conserved densities of the whole hierarchy (2.5) and therefore they are in involution in pairs.

Proof. A direction calculation leads to

$$\left\langle V_\lambda, \frac{\partial U}{\partial \lambda}\right\rangle = \frac{1}{2}G_\lambda, \quad \left\langle V_\lambda, \frac{\partial U}{\partial v}\right\rangle = -\frac{1}{2}G_\lambda.$$

By using the trace identity (2.12) and the expansion (2.13), we obtain

$$\frac{\delta}{\delta v}\left(\sum_{m=-\infty}^{-1} G_m\lambda^{-m}\right) = \sum_{m=-\infty}^{-1}(m-1-\beta)G_{m-1}\lambda^{-m} + (-1-\beta)G_{-1}, \tag{2.16}$$

$m = -1, -2\cdots.$

If taking $G_{-1} \neq 0$, form (2.16) we find $\beta = -1$ and

$$\frac{\delta H_m}{\delta v} = G_{m-1}, \quad m = -1, -2\cdots, \tag{2.17}$$

where H_m are given by (2.15). Substituting (2.17) into (2.5) yields the bi-Hamiltonian structures (2.14).

Next, we consider infinitely many conserved densities to guarantee integrability of the hierarchy (2.16). Since J and K are skew-symmetric operators, we infer that

$$\mathcal{L}^*J = (J^{-1}K)^*J = -K^* = K = J\mathcal{L},$$

which implies

$$\{H_n, H_m\} = \left(\frac{\delta H_n}{\delta v}, J\frac{\delta H_m}{\delta v}\right) = (\mathcal{L}^n G_{-1}, J\mathcal{L}^m G_{-1}) = (\mathcal{L}^n G_{-1}, \mathcal{L}^* J\mathcal{L}^{m-1} G_{-1})$$
$$= (\mathcal{L}^{n+1} G_{-1}, J\mathcal{L}^{m-1} G_0) = \{H_{n+1}, H_{m-1}\}, \quad m, n \leq -1.$$

Repeating the above argument gives

$$\{H_n, H_m\} = \{H_m, H_n\} = \{H_{m+n}, H_{-1}\}. \tag{2.18}$$

On the other hand, we find

$$\{H_m, H_n\} = (\mathcal{L}^m G_{-1}, J\mathcal{L}^n G_{-1}) = (J^*\mathcal{L}^m G_{-1}, \mathcal{L}^n G_{-1}) = -\{H_n, H_m\}. \tag{2.19}$$

Then combining (2.18) with (2.19) leads to

$$\{H_m, H_n\} = 0,$$

which implies that $\{H_m\}$ are in involution, and therefore the hierarchy (2.14) are integrable in Liouville sense.

Especially, under the constraint (1.3), we obtain bi-Hamilton structures of the NKdV equation (1.2)

$$v_t = K\frac{\delta H_{-1}}{\delta v} = J\frac{\delta H_0}{\delta u},$$

where two Hamiltonian functions are given by

$$H_0 = \frac{1}{3}u^3, \quad H_{-1} = -u^2,$$

which can also be written in a conserved density form in the sense of equivalence class

$$H_0 \sim -\frac{1}{3}\int u^3 dx, \quad H_{-1} \sim -\int u^2 dx.$$

3. Relations to other important equations

3.1. Kupershmidt deformation

Recently a class of new integrable systems, known as the Kupershmidt deformation of soliton equations, have attracted much attention. This topic starts from Kupershmidt, Karasu-Kalkani' work [27, 37, 39].

For the Lenard operator pair (2.6), we define Lenard gradients recursively by

$$KG_j = JG_{j+1}, \; KG_{-1} = JG_0 = 0, \quad j = 0, \pm 1, \pm 2, \cdots,$$

then KdV hierarchy is

$$v_t = KG_{m-1} = JG_m, \quad m = 0, \pm 1, \pm 2, \cdots \tag{3.1}$$

which contains both the NKdV hierarchy and the positive order KdV hierarchy.

The first equation ($m = 0$) in the KdV hierarchy (3.1) is trivial system

$$v_t = JG_0 = 0, \quad KG_{-1} = JG_0 = 0, \tag{3.2}$$

which can be regarded as is a "sharp threshold" equation of the NKdV hierarchy and positive order KdV hierarchy.

A Kupershmidt nonholonomic deformation of the hierarchy (3.1) takes

$$\begin{aligned} &v_t = JG_m + Jw, \quad m = 0, \pm 1, \pm 2, \cdots, \\ &Kw = 0, \end{aligned} \tag{3.3}$$

where two operators K and J are given by (1.4). Then the first flow ($m = 0$) of the hierarchy (3.3) is exactly the NKdV equation (1.1)

$$\begin{aligned} &v_t = w_x, \\ &w_{xxx} + 4vw_x + 2v_x w = 0, \end{aligned}$$

which may be regarded as a Kupershmidt nonholonomic deformation of the threshold equation (3.2)

3.2. NKdV hierarchy with self-consistent sources

Soliton equations with self-consistent sources have important physical applications, for example, the KdV equation with self-consistent source describes the interaction of long and short capillary-gravity waves [44, 50, 51, 52].

For the N distinct λ_j of the spectral problem (2.1), the functional gradient of λ_j with respect to v is

$$\frac{\delta \lambda_j}{\delta v} = \psi_j^2.$$

Here we define the whole KdV hierarchy with self-consistent sources as follows

$$\begin{aligned} &v_t = JG_m + \alpha J \frac{\delta \lambda}{\delta v} = JG_m + \alpha J \sum_{j=1}^{N} \psi_j^2, \\ &\psi_{j,xx} + (v + \lambda_j)\psi_j = 0, \\ &m = 0, \pm 1, \pm 2, \cdots; \ j = 1, \cdots, N. \end{aligned} \tag{3.4}$$

Taking $m = 1$ in the hierarchy (3.4) leads to the KdV equation with self-consistent sources

$$\begin{aligned}
&v_t = \frac{1}{4}(v_{xxx} + 6vv_x) + \alpha\partial_x \sum_{j=1}^{N} \psi_j^2,\\
&\psi_{j,xx} + (v + \lambda_j)\psi_j = 0,\ \ j = 1, \cdots, N,
\end{aligned}$$

while choosing $m = -1$ in the hierarchy (3.4) gives the NKdV equation with self-consistent sources

$$\begin{aligned}
&v_t = w_x + \alpha\partial_x \sum_{j=1}^{N} \psi_j^2,\\
&w_{xxx} + 4vw_x + 2v_x w = 0,\\
&\psi_{j,xx} + (v + \lambda_j)\psi_j = 0,\ \ j = 1, \cdots, N.
\end{aligned}$$

Obviously, taking $N = 1,\ m = 0,\ \alpha = 1,\ v \to v + \lambda_1$ in the hierarchy (3.4), then we get the NKdV equation (1.2)

$$v_t = (\psi_1^2)_x, \quad \psi_{1,xx} + v\psi_1 = 0,$$

which may be regarded as the threshold equation (3.2) with self-consistent sources.

3.3. Reduction of the NKdV equation (1.1)

Theorem 3. (u, v) is a solution of NKdV-1 equation (1.2) if and only if (w, v) with $w = u^2$ is a solution of NKdV equation (1.1) under the transformation

$$u_{xx} + vu = 0, \tag{3.5}$$

which is actually a linear Schrödinger equation or Hill equation.

Proof. Let

$$w = u^2, \tag{3.6}$$

then by (1.1), we have

$$v_t = w_x = 2uu_x,$$

which is the first equation of (1.2). By (3.6), the second equation of (1.1) leads to

$$3u_x(u_{xx} + vu) + u(u_{xx} + vu)_x = 0,$$

or equivalently,

$$[u^3(u_{xx}+vu)]_x=0, \tag{3.7}$$

Apparently, according to (3.7), if (u,v) is a solution of the NKdV-1 equation (1.2), then (w,v) is a solution of the NKdV equation (1.1) where $w=u^2$. Reversely, if (w,v) is a solution of the NKdV equation (1.1), then (u,v) is also a solution of the NKdV-1 equation (1.2) under the transformation (3.5).

For a given function ϕ, let us define the following Baker-Akhiezer function

$$u=\exp\left(\int_0^x \phi dx\right), \tag{3.8}$$

then (3.2) yields the following Riccati equation

$$\phi_x+\phi^2+v=0. \tag{3.9}$$

So, we have

Theorem 4. (u,v) is a solution of the NKdV-1 equation (1.2) if and only if (w,v) is a solution of the NKdV equation (1.1) as ϕ is a solution of the Riccati equation (3.9) while u is the Baker-Akhiezer function (3.8) and $w=u^2$.

3.4. Ermakov-Pinney equation

The Ermakov-Ray-Reid systems

$$\begin{aligned}\psi_{xx}+\omega^2(x)\psi&=\frac{1}{\psi^2\phi}F(\frac{\phi}{\psi}),\\ \phi_{xx}+\omega^2(x)\phi&=\frac{1}{\psi\phi^2}G(\frac{\psi}{\phi}),\end{aligned}$$

were originally introduced by Ermakov [12, 70]. Due to their nice mathematical properties of Ermakov systems admitting a novel integral of motion together with a concomitant nonlinear superposition principle and extensively physical applications, there has been numerous an extensive literature devoting to the analysis of the Ermakov systems [2, 60, 71, 72, 73, 74]. The most simple case is equation

$$\psi_{xx}+\omega^2(x)\psi=\frac{c}{\psi^3},$$

which is called the Ermakov-Pinney equation. The Ermakov-Pinney equation is a quite famous example of a nonlinear ordinary differential equation. Such an equation (and generalizations thereof) have been shown to be relevant to a number of physical contexts including quantum cosmology, quantum field theory, nonlinear elasticity and nonlinear optics [16, 75, 79]. A recent account of some of its properties along with applications in cosmological settings can be found in Ref. [69].

Proposition 4. Suppose that (w, v) is a solution of the NKdV equation (1.1). Let

$$w = p_t = \psi^2,\ v = p_x,$$

then ψ satisfies a Ermakov-Pinney equation

$$\psi_{xx} + v\psi = \frac{\mu}{\psi^3}, \tag{3.9}$$

where μ is an integration constant.

Especially, if (u, v) is the solution of the NKdV-1 equation (1.2), let

$$u = \phi \exp\left(i \int \mu\phi^{-2} dx\right), \tag{3.10}$$

then ϕ satisfies the Ermakov-Pinney equation

$$\phi_{xx} + v\phi = \frac{\mu}{\phi^3}. \tag{3.11}$$

Proof. Substituting transformation $w = \psi^2$ into the second equation of the NKdV equation (1.1) yields

$$w_{xxx} + 4vw_x + 2v_x w = 2\psi(\psi_{xx} + v\psi)_x + 6\psi_x(\psi_{xx} + v\psi)$$
$$= \frac{2}{\psi^2}[(\psi_{xx} + v\psi)\psi^3]_x = 0,$$

which leads to (3.9).

Substituting transformation (3.10) into the second equation of the NKdV equation (1.2) yields

$$u_x + vu = (\phi_{xx} + v\phi - \frac{\mu}{\phi^3}) \exp\left(i \int \mu\phi^{-2} dx\right) = 0,$$

which implies (3.11).

Proposition 5. For given function v, let ψ_1, ψ_2 are two solutions of linear Schrödinger equation

$$u_{xx} + vu = 0, \tag{3.12}$$

then equation

$$w_{xxx} + 4w_x v + 2v_x w = 0 \tag{3.13}$$

admits a general solution

$$w = a\psi_1^2 + 2b\psi_1\psi_2 + c\psi_2^2, \tag{3.14}$$

where

$$ac - b^2 = \frac{\mu}{2W}, \quad W = \psi_1\psi_{2,x} - \psi_{1,x}\psi_2.$$

Proof. Let $w = \psi^2$, by proposition 4, then ψ satisfies a Ermakov-Pinney equation (3.9). It is easy to check that if ψ_1 and ψ_2 are two solutions of equation (3.14), then

$$\psi = \sqrt{a\psi_1^2 + 2b\psi_1\psi_2 + c\psi_2^2}$$

is a solution of equation (3.9). So (3.14) is a general solution of the equation (3.13).

4. Darboux transformation of NKdV equations

In this section, we shall construct a Darboux transformation for general NKdV equation (1.1), and then reduce it to the NKdV-1 equation (1.2).

4.1. Darboux transformation

A Darboux transformation is actually a special gauge transformation

$$\tilde{\psi} = T\psi \tag{4.1}$$

of solutions of the Lax pair (2.9), here T is a differential operator (For the Lax pair (2.10), the Darboux transformation with $\lambda = 0$ can be obtained). It requires that $\tilde{\psi}$ also satisfies the same Lax pair (2.9) with some $\tilde{L}$ and $\tilde{M}$, i. e.

$$\begin{aligned} \tilde{L}\tilde{\psi} &= \lambda\tilde{\psi}, \qquad \tilde{L} = TLT^{-1}, \\ \tilde{M}\tilde{\psi} &= 0, \qquad \tilde{M} = TMT^{-1} \end{aligned} \tag{4.2}$$

Apparently, we have

$$[\tilde{L}, \tilde{M}] = T[L, M]T^{-1},$$

which implies that $\tilde{L}$ and $\tilde{M}$ are required to have the same forms as L and M, respectively, in order to make system (2.9) invariant under the gauge transformation (3.4). At the same time the old potentials u and v in L, M will be mapped into new potentials $\tilde{u}$ and $\tilde{v}$ in $\tilde{L}$, $\tilde{M}$. This process can be done continually and usually it may yield a series of multi-soliton solutions.

Let us now set up a Darboux transformation for the system (2.9). Let $\psi_0 = \psi_0(x, t)$ be a basic solution of Lax pair (2.9) for λ_0, and use it to define the following gauge transformation

$$\tilde{\psi} = T\psi, \tag{4.3}$$

where

$$T = \partial_x - \sigma, \quad \sigma = \partial_x \ln \psi_0. \tag{4.4}$$

From (2.9) and (4.4), one can see that σ satisfies

$$\sigma_x + \sigma^2 + v - \lambda = 0 \tag{4.5}$$

$$4\sigma_{xxt} + 12\sigma_x\sigma_t + 4v\sigma_t + 2w\sigma_x + 6\sigma\sigma_{xt} + 3w_{xx} = 0. \tag{4.6}$$

Proposition 6. The operator $\tilde{L}$ determined by (4.2) has the same form as L, that is,

$$\tilde{L} = \partial_x^2 + \tilde{v},$$

where the transformation between v and $\tilde{v}$ is given by

$$\tilde{v} = v + 2\sigma_x. \tag{4.7}$$

The transformation: $(\psi, v) \to (\tilde{\psi}, \tilde{v})$ is called a Darboux transformation of the first spectral problem of Lax pair (2.9).

Proof. According to (4.2), we just prove

$$\tilde{L}T = TL,$$

that is,

$$(\partial_x^2 + \tilde{v})(\partial_x - \sigma) = (\partial_x - \sigma)(\partial_x^2 + v),$$

which is true through (4.5) and (4.7).

Proposition 7. Under the transformation (4.3), the operator $\tilde{M}$ determined by (4.2) has the same form as M, that is,

$$\tilde{M} = 4\partial_x^2\partial_t + 4\tilde{v}\partial_t - 2\tilde{w}\partial_x - 3\tilde{w}_x, \tag{4.8}$$

where the transformations between w, v and $\tilde{w}$, $\tilde{v}$ are given by

$$\tilde{w} = w + 2\sigma_t, \quad \tilde{v} = v + 2\sigma_x. \tag{4.9}$$

The transformation: $(\psi, w, v) \to (\tilde{\psi}, \tilde{w}, \tilde{v})$ is Darboux transformation of the second spectral problem of Lax pair (2.9).

Proof. To see that $\tilde{M}$ has the form (4.8) same as M, we just prove

$$\tilde{M}T = TM, \tag{4.10}$$

where

$$\tilde{M} = 4\partial_x^2\partial_t + f\partial_t + g\partial_x + h, \tag{4.11}$$

with three functions f, g, and h to be determined. Substituting $\tilde{M}$, M, L into (4.10) and comparing the coefficients of all distinct operators lead to:

coefficient of operator $\partial_x\partial_t$

$$f = 4v + 8\sigma_x = 4\tilde{v},$$

which holds by using (4.9).

coefficient of operator ∂_x^2

$$g = 2w + 4\sigma_t = 2\tilde{w},$$

which implies from (4.9).

coefficient of operator ∂_x

$$\begin{aligned} h &= 8\sigma_{xt} + 5w_x - 2\sigma w + g\sigma = 6\sigma_{xt} + 3w_x + 2(\sigma_x + \sigma^2 + v)_t \\ &= 6\sigma_{xt} + 3w_x = 3\tilde{w}_x, \end{aligned}$$

here we have used equation (4.5) and (4.9).

coefficient of operator ∂_t

$$-4\sigma_{xx} - f\sigma = 4v_x - 4v\sigma,$$

that is,

$$\sigma_{xx} + 2\sigma\sigma_x + v_x = 0.$$

which holds by using (4.5).

coefficient of non-operator:

$$4\sigma_{xxt}+f\sigma_t+g\sigma_x+\sigma h+3w_{xx}-3\sigma w_x=0,$$

that is,

$$4\sigma_{xxt}+12\sigma_x\sigma_t+4v\sigma_t+2w\sigma_x+6\sigma\sigma_{xt}+3w_{xx}=0,$$

which is the equation (4.6). We complete the proof. $\square$

Propositions 4 and 5 tell us that the transformations (4.3) and (4.9) send the Lax pair (2.9) to another Lax pair (4.2) in the same type. Therefore, both of the Lax pairs lead to the same NKdV equation (1.1). So, we call the transformation $(\psi,w,v)\to(\tilde{\psi},\tilde{w},\tilde{v})$ a Darboux transformation of the NKdV equation (1.1). In summary, we arrive at the following theorem.

Theorem 5. A solution $w,\ v$ of the NKdV equation (1.1) is mapped into its new solution $\tilde{w},\ \tilde{v}$ under the Darboux transformations (4.3) and (4.9).

4.2. Reduction of Darboux transformations

To get Darboux transformations for the NKdV-1 equation (1.2), we consider two reductions of the Darboux transformations (4.3) and (4.9).

Corollary 1. Let $\lambda=k^2>0$, then under the constraints $w=u^2,\ v=-u_{xx}/u$, the Darboux transformations (4.3) and (4.9) are reduced to a Darboux transformation of the NKdV-1 equation (1.2) $(\psi,v,u)\to(\tilde{\psi},\tilde{v},\tilde{u})$, where

$$\tilde{\psi}=T\psi,\quad \tilde{v}=v+2\sigma_x,\quad \tilde{u}=k^{-1}(u_x-\sigma u)=k^{-1}Tu. \tag{4.13}$$

Proof. For $\lambda>0$, suppose that (v,u) is a solution of the NKdV-1 equation and ψ is an eigenfunction of the Lax pair (1.4), then we have

$$\lambda^{-1}(u\psi_x-u_x\psi)=\partial_x^{-1}(u\psi).$$

Therefore, the Lax pair (1.4) can be rewritten as

$$\begin{aligned}&\psi_{xx}+v\psi=\lambda\psi,\\ &\psi_t=\frac{1}{2}u\lambda^{-1}(u\psi_x-u_x\psi)=\frac{1}{2}u\partial_x^{-1}(u\psi)=N(u,\lambda)\psi,\end{aligned}\tag{4.12}$$

where $N=N(u,\lambda)=\frac{1}{2}u\partial_x^{-1}u$.

According to Proposition 6, the spectral problem of Lax pair (4.12) is covariant under the transformation (4.13), that is,

$$\tilde{\psi}_{xx} + \tilde{v}\tilde{\psi} = \lambda\tilde{\psi}.$$

So, we only need to prove

$$\tilde{\psi}_t = N(\tilde{u}, \lambda)\tilde{\psi} \tag{4.14}$$

Substituting (4.13) into the left hand side of (4.14) yields

$$\begin{aligned}\tilde{\psi}_t &= (\psi_t)_x - (\sigma\psi)_t = (N\psi)_x - \sigma N\psi - (\psi_0^{-1}N\psi_0)_x\psi,\\ &= \frac{1}{2}[(u_x - \sigma u)\partial_x^{-1}(u\psi) - \psi_0^{-1}\psi(u_x - \sigma u)\partial_x^{-1}(u\psi_0)]\\ &= \frac{1}{2}k\tilde{u}[\partial_x^{-1}(u\psi) + k^{-2}(u_x - \sigma u)\psi].\end{aligned} \tag{4.15}$$

In the same way, substituting (4.13) into the right hand side of (4.14) produces

$$\begin{aligned}N(\tilde{u}, \lambda)\tilde{\psi} &= \frac{1}{2}\tilde{u}\partial_x^{-1}[k^{-1}(u_x - \sigma u)(\psi_x - \sigma\psi)]\\ &= \frac{1}{2}k^{-1}\tilde{u}[u_x\psi - \partial_x^{-1}(u_{xx}\psi) - \sigma u\psi + \partial_x^{-1}(\psi_0^{-1}\psi_{0,xx}u\psi)]\\ &= \frac{1}{2}k^{-1}\tilde{u}[k^2\partial_x^{-1}(u\psi) + (u_x - \sigma u)\psi].\end{aligned} \tag{4.16}$$

Combining (4.15) with (4.16) implies that (4.14) holds. □

In a similar way, we also have the following result.

Corollary 2. Let $\lambda = 0$, then under the constraints $w = u^2,\ v = -u_{xx}/u$, the Darboux transformations (4.3) and (4.9) are reduced to another Darboux transformation of the NKdV-1 equation (1.2) $(\psi, v, u) \to (\tilde{\psi}, \tilde{v}, \tilde{u})$, where

$$\begin{aligned}&\tilde{v} = v + 2\sigma_x, \quad \tilde{\psi} = \psi - \psi_0^{-1}\sigma\partial_x^{-1}(\psi_0\psi),\\ &\tilde{u} = \begin{cases} \psi_0^{-1}\sigma, & u = 0,\\ u - \psi_0^{-1}\sigma\partial_x^{-1}(\psi_0 u), & u \neq 0.\end{cases}\end{aligned} \tag{4.17}$$

with $\sigma = \partial_x \ln(1 + \partial_x^{-1}\psi_0^2)$.

5. Applications of the Darboux transformation

In this section, we shall apply the Darboux transformations (4.3) and (4.9) to obtain kink-type and bell-type of explicit solutions for the NKdV equation (1.1).

5.1. The kink-wave solutions

For the case of $\lambda = k^2 > 0$, we substitute $v = 0, w = 1$ into the Lax pair (2.9) and choose the following basic solution

$$\psi = e^{\xi} + e^{-\xi} = 2\cosh\xi, \quad \xi = kx - \frac{1}{2k}t + \gamma, \tag{5.1}$$

where γ and k are two arbitrary constants.

Taking $\lambda = k_1^2$, then (4.4) and (5.1) lead to

$$\sigma_1 = \partial_x \ln\psi = k_1 \tanh\xi_1, \quad \xi_1 = k_1 x - \frac{1}{2k_1}t + \gamma_1.$$

The Darboux transformation (4.9) gives bell-type solution for the NKdV equation (1.1)

$$\begin{aligned} \tilde{v}^I &= 2\sigma_{1,x} = 2k_1^2 \mathrm{sech}^2\xi_1, \\ \tilde{w}^I &= 1 - 2\sigma_{1,t} = \tanh^2\xi_1. \end{aligned} \tag{5.2}$$

By using Darboux trasformation (4.13), we get a kink-type wave solution for the NKdV equation (1.2)

$$\tilde{u}^I = k_1^{-1}(u_x - \sigma u) = -\tanh\xi_1, \quad \xi_1 = k_1 x - \frac{1}{2k_1}t + \gamma_1. \tag{5.3}$$

Remark 1. There is much difference between traveling waves of the NKdV equation (1.2) and of the classical KdV equation. For the NKdV equation (1.2), its one-wave solution is a negative-moving (i.e. from right to left) kink-wave with velocity $-1/2k_1^2$, amplitude ± 1 and width $1/k_1$. Its amplitude is independent of velocity, and width is directly proportional to the velocity. For the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{5.4}$$

one-soliton solution is

$$u = \frac{k^2}{2}\mathrm{sech}^2\frac{k(x - k^2 t)}{2}, \tag{5.5}$$

which is a bell-type positive-moving wave with velocity k^2, amplitude $k^2/2$ and width $1/k$, respectively. Its amplitude is directly proportional to velocity, and width is inversely proportional to the velocity.

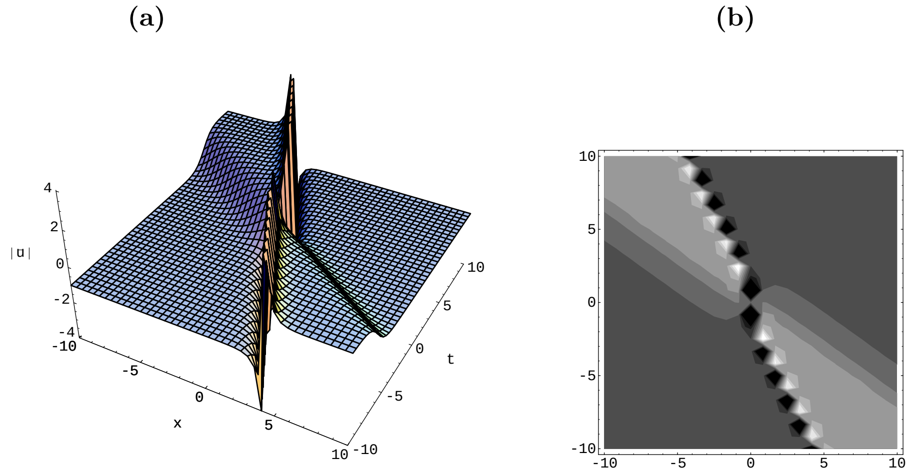

FIGURE 1. The two-kink wave solution $u(x,t)$ with parameters: $k_1 = 1$, $k_2 = 0.6$. (a) Perspective view of the wave. (b) Overhead view of the wave, with contour plot shown. The bright lines are crests and the dark lines are troughs.

Let us now construct two-kink solutions to see the interaction of two kink solutions. According to (4.4),

$$\tilde{\psi} = T\psi = (\partial_x - \sigma_1)(e^{\xi} + e^{-\xi}) \tag{5.6}$$

is also an eigenfunction of Lax pair (2.9). Taking $\lambda = k_2^2$, we have

$$\sigma_2 = -k_1 \tanh \xi_1 + \frac{k_1^2 - k_2^2}{k_1 \tanh \xi_1 - k_2 \tanh \xi_2}. \tag{5.7}$$

Repeating the Darboux transformation (4.9) one more time, we get two soliton solution for the the NKdV equation (1.1)

$$\tilde{v}^{II} = \tilde{v}^{I} + 2\sigma_{2,x} = \frac{(k_1^2 - k_2^2)(k_2^2 \mathrm{sech}^2 \xi_2 - k_1^2 \mathrm{sech}^2 \xi_1)}{(k_1 \tanh \xi_1 - k_2 \tanh \xi_2)^2},$$

$$\tilde{w}^{II} = \tilde{w}^{I} - 2\sigma_{2,t} = \left(\frac{k_1 \tanh \xi_2 - k_2 \tanh \xi_1}{k_1 \tanh \xi_1 - k_2 \tanh \xi_2} \right)^2.$$

Therefore, we obtain a two-kink wave solution of the NKdV equation (1.2)

$$\tilde{\tilde{u}} = \frac{k_2 \tanh \xi_1 - k_1 \tanh \xi_2}{k_1 \tanh \xi_1 - k_2 \tanh \xi_2}. \tag{5.8}$$

Let us use the two-kink wave solution (5.8) to analyze interaction of the two one-soliton solutions. Without loss of generality, we suppose $k_1 > k_2 > 0$, then

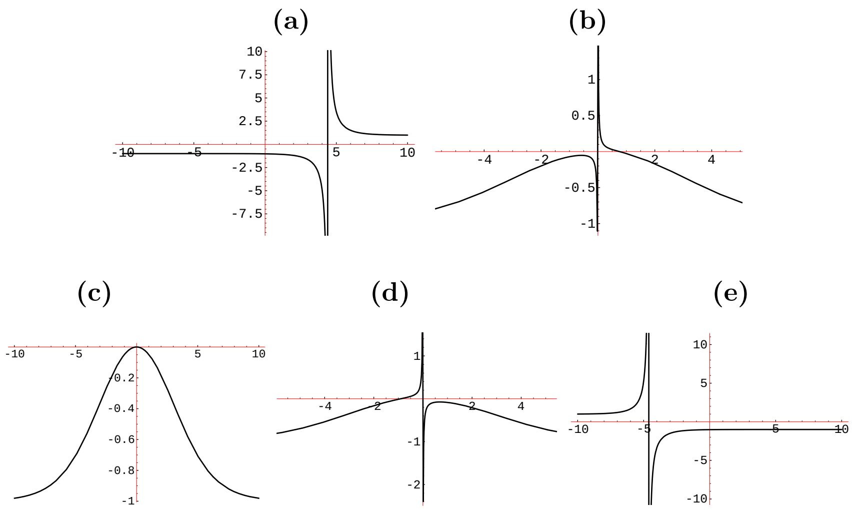

FIGURE 2. Interaction between singular soliton cschξ_1 and smooth soliton sechξ_2 with parameters: (a) $t=-3$, (b) $t=-0.05$, (c) $t=0$, (d) $t=0.05$, (e) $t=3$.

we have

$$\xi_2 = \frac{k_2}{k_1}\left[\xi_1 - \frac{k_1}{2}(\frac{1}{k_2^2} - \frac{1}{k_1^2})t\right].$$

Therefore, on the fixed line ξ_1 =constant, we get

$$\tanh \xi_2 \sim -1, \quad t \to +\infty,$$

and it follows (5.8) that

$$\tilde{\tilde{u}} \sim \frac{k_2 \tanh \xi_1 + k_1}{k_1 \tanh \xi_1 + k_2} = \coth\left(\xi_1 - \frac{1}{2}\ln\frac{k_1-k_2}{k_1+k_2}\right), \ t \to +\infty. \tag{5.9}$$

In a similar way, one can get

$$\tanh \xi_2 \sim 1 \ \text{ as } t \to -\infty,$$

which are main parts compared with terms 1 and $e^{2\xi_1}$, and it follows (3.19) that

$$\tilde{\tilde{u}} \sim \frac{k_2 e^{2\xi_1} - k_1}{k_1 e^{2\xi_1} - k_2} = \coth\left(\xi_1 + \frac{1}{2}\ln\frac{k_1-k_2}{k_1+k_2}\right), \ t \to -\infty. \tag{5.10}$$

In a similar way, on the line ξ_2 =constant, we will arrive at

$$\tilde{\tilde{u}} \sim \tanh\left(\xi_2 + \frac{1}{2}\ln\frac{k_1-k_2}{k_1+k_2}\right), \quad \text{as} \quad t \to +\infty, \tag{5.11}$$

$$\tilde{\tilde{u}} \sim \tanh\left(\xi_2 - \frac{1}{2}\ln\frac{k_1-k_2}{k_1+k_2}\right), \quad \text{as} \quad t \to -\infty. \tag{5.12}$$

Remark 2. From expressions (5.9)-(5.12), we see that the two-kink wave solution (5.8) is a singular solution, which is able to be decomposed into a singular kink-type solution and a smooth kink wave solutions. The expressions (5.10) and (5.12) show that the wave $\tanh\xi_2$ is on the left of the wave $\coth\xi_1$ before their interaction, while the expressions (5.9) and (5.11) show that the wave $\coth\xi_1$ is on the left of the wave $\tanh\xi_2$ after their interaction. The shapes of the two kink waves $\coth\xi_1$ and $\tanh\xi_2$ don't change except their phases. Their phases of the two waves $\coth\xi_1$ and $\tanh\xi_2$ are $\ln\frac{k_1-k_2}{k_1+k_2}>0$ and $-\ln\frac{k_1-k_2}{k_1+k_2}<0$, respectively as the wave is negatively going along the $x-$axis. Very interesting case is particular at $t=0$: collision of such two kink waves forms a smooth bell-type soliton and its singularity disappears (See Figure 2).

After their interaction It can be seen that the the two kink waves resume their original shapes. At the right moment of interaction, the two kink waves are fused into a smooth bell-type soliton. The two-kink wave interactions possess the regular elastic-collision features and pass through each other, and their shapes keep unchanged with a phase shift after the interaction. Here, we also demonstrate a fact that the large-amplitude kink wave with faster velocity overtakes the small-amplitude one, after collision, the smaller one is left behind.

5.2. The bell-type soliton solutions

(i) For the case of $\lambda=0$ (i.e. without parameter λ), we substitute $v=-k^2,\ w=0$ into the Lax pair (2.10), and choose the following basic solution as

$$\psi=e^{\xi}+e^{-\xi},\quad \xi=kx+\frac{1}{2k}t,$$

where k is an arbitrary constant.

Taking $k=k_1$, (4.4) gives

$$\sigma=\sigma_1=\partial_x\ln\psi=k_1\tanh\xi_1,\quad \xi_1=k_1x+\frac{1}{2k_1}t. \tag{5.13}$$

Using the Darboux transformation (4.9), we have one-soliton solution for the NKdV equation (1.1)

$$\begin{aligned} \tilde{v} &= v + 2\sigma_{1,x} = 2k_1^2 \mathrm{sech}^2 \xi_1 - k_1^2, \\ \tilde{w} &= -2\sigma_{1,t} = \mathrm{sech}^2 \xi_1. \end{aligned} \tag{5.14}$$

So, we get a one-soliton solution for the NKdV-1 equation (1.2) by using Darboux transformation (4.17)

$$\tilde{u} = \mathrm{sech}\xi_1, \quad \xi_1 = k_1 x + \frac{1}{2k_1} t. \tag{5.15}$$

Remark 3. For the negative order KdV equation (1.2), its one-soiton solution (5.15) is a smooth bell-type negative-moving wave, whose velocity, amplitude and width are $1/2k_1^2$, ± 1 and $1/k_1$, respectively. Its amplitude is independent of velocity, and width is directly proportional to the velocity.

(ii) For the case of $\lambda = -k^2$, we take a seed solution of $v = -2k^2, w = 1$ in the Lax pair (2.9), and choose the following basic solution as

$$\psi = e^{\xi} + e^{-\xi}, \quad \xi = kx - \frac{1}{2k} t + \gamma,$$

where k is an arbitrary constant.

Taking $k = k_1$ sends (3.7) to

$$\sigma = \sigma_1 = \partial_x \ln \psi = k_1 \tanh \xi_1, \quad \xi_1 = k_1 x - \frac{1}{2k_1} t + \gamma_1. \tag{5.13}$$

Using the Darboux transformation (3.12), we then get one-soliton solution

$$\begin{aligned} \tilde{v}^I &= v + 2\sigma_{1,x} = -2k_1^2 \mathrm{tanh}^2 \xi_1 + \gamma_1, \\ \tilde{w}^I &= 1 - 2\sigma_{1,t} = 1 + \mathrm{sech}^2 \xi_1, \end{aligned} \tag{5.14}$$

which cannot satisfies the constraint (3.3), so $\sqrt{\tilde{w}^I}$ is not soliton for the NKdV equation (1.2).

Remark 4. For the NKdV equation (1.1), its one-soiton solution (5.14) is a smooth bell-type positive-moving wave, whose velocity, amplitude and width are $1/2k_1^2$, ± 1 and $1/k_1$, respectively. Its amplitude is independent of velocity, and width is directly proportional to the velocity.

Let's construct a two-soliton solution of the NKdV equation (1.1). According to the gauge transformation (4.4),

$$\tilde{\psi} = T\psi = (\partial_x - \sigma_1)(e^{\xi} + e^{-\xi})$$

is also an eigenfunction of Lax (2.9). We have

$$\sigma_2 = -k_1 \tanh \xi_1 + \frac{k_1^2 - k_2^2}{k_1 \tanh \xi_1 - k_2 \tanh \xi_2}.$$

Repeating the Darboux transformation (4.9) one more time, we obtain

$$\tilde{v}^{II} = \tilde{v}^{I} + 2\sigma_{2,x} = \frac{(k_1^2 - k_2^2)(k_2^2 \mathrm{sech}^2 \xi_2 - k_1^2 \mathrm{sech}^2 \xi_1)}{(k_1 \tanh \xi_1 - k_2 \tanh \xi_2)^2},$$

$$\tilde{w}^{II} = \tilde{w}^{I} - 2\sigma_{2,t} = \left(\frac{k_1 \tanh \xi_2 - k_2 \tanh \xi_1}{k_1 \tanh \xi_1 - k_2 \tanh \xi_2}\right)^2,$$

which is the same for NKdV-1 equation (1.2). So we get two-soliton solution with (5.8)

$$\tilde{\tilde{u}} = \pm \frac{k_1 \tanh \xi_2 - k_2 \tanh \xi_1}{k_1 \tanh \xi_1 - k_2 \tanh \xi_2},$$

but here $\xi_j = k_j x - \frac{1}{2k_j} t,\ j = 1, 2$.

6. Bilinearization of the NKdV equation

The bilinear derivative method, developed by Hirota [28], has become a powerful approach to construct exact solutions of nonlinear equations. Once a nonlinear equation is written in a bilinear form by using some transformation, then multi-solitary wave solutions or quasi-periodic wave solutions can usually be obtained [29, 30, 34, 35, 45, 84, 13]. However, unfortunately, this method is not as direct as many people might wish because the original equation is reduced to two or more bilinear equations under new variables called Hirota's variables. Since no a general rule to select Hirota's variables, there is no rule to choose some essential formulas (such as exchange formulas), either. Especially the construction of bilinear Bäcklund transformation relies on a particular skill and appropriate exchange formulas. On the other hand, in recent years Lambert and his co-workers have found a kind of the generalized Bell polynomials playing important role in seeking the characterization of bilinearized equations. Based on the Bell polynomials, they presented an alternative procedure to obtain parameter families of a

bilinear Bäcklund transformation and Lax pairs for soliton equations in a quick and short way [26, 40, 41].

6.1. Multi-dimensional binary Bell polynomials

The main tool we use here is a class of generalized multi-dimensional binary Bell polynomials [26]-[41].

Definition 1. Let $n_k \geq 0,\ k = 1, \cdots, \ell$ denote arbitrary integers, $f = f(x_1, \cdots, x_\ell)$ be a C^∞ multi-variable function, then

$$Y_{n_1x_1,\cdots,n_\ell x_\ell}(f) \equiv \exp(-f)\partial_{x_1}^{n_1}\cdots\partial_{x_\ell}^{n_\ell}\exp(f) \tag{6.1}$$

is a polynomial in the partial derivatives of f with respect to $x_1, \cdots, x_\ell$, which we call a multi-dimensional Bell polynomial (a generalized Bell polynomial or Y-polynomial).

For the two dimensional case, let $f = f(x,t)$, then the associated Bell polynomials through (6.1) can produce the following representatives:

$$Y_x(f) = f_x,\ Y_{2x}(f) = f_{2x} + f_x^2, \quad Y_{3x}(f) = f_{3x} + 3f_xf_{2x} + f_x^3,$$
$$Y_{x,t}(f) = f_{x,t} + f_xf_t, \quad Y_{2x,t}(f) = f_{2x,t} + f_{2x}f_t + 2f_{x,t}f_x + f_x^2f_t, \cdots .$$

Definition 2. Based on the use of above Bell polynomials (6.21), the multi-dimensional binary Bell polynomials ($\mathcal{Y}$-polynomials) are defined by

$$\mathcal{Y}_{n_1x_1,\cdots,n_\ell x_\ell}(g,h) = Y_{n_1x_1,\cdots,n_\ell x_\ell}(f)\ \Big|_{f_{r_1x_1,\cdots,r_\ell x_\ell}=\begin{cases} g_{r_1x_1,\cdots,r_\ell x_\ell}, & r_1+\cdots+r_\ell \ \text{ is odd}, \\ h_{r_1x_1,\cdots,r_\ell x_\ell}, & r_1+\cdots+r_\ell \ \text{ is even}, \end{cases}}$$

which is a multi-variable polynomial with respect to all partial derivatives $g_{r_1x_1,\cdots,r_\ell x_\ell}$ ($r_1+\cdots+r_\ell$ is odd) and $h_{r_1x_1,\cdots,r_\ell x_\ell}$ ($r_1+\cdots+r_\ell$ is even), $r_k = 0, \cdots, n_k,\ k = 0, \cdots, \ell$.

The binary Bell polynomials also inherit partial structures of the Bell polynomials. The first few lower order binary Bell Polynomials are

$$\begin{aligned} &\mathcal{Y}_x(g) = g_x,\ \mathcal{Y}_{2x}(g,h) = h_{2x} + g_x^2, \ \ \mathcal{Y}_{x,t}(g,h) = h_{xt} + g_xg_t. \\ &\mathcal{Y}_{3x}(g,h) = g_{3x} + 3g_xh_{2x} + g_x^3, \cdots . \end{aligned} \tag{6.2}$$

Proposition 8. The link between binary Bell polynomials $\mathcal{Y}_{n_1x_1,\cdots,n_\ell x_\ell}(g,h)$ and the standard Hirota bilinear expression $D_{x_1}^{n_1}\cdots D_{x_\ell}^{n_\ell}F\cdot G$ can be given by an

identity

$$\mathcal{Y}_{n_1x_1,\cdots,n_\ell x_\ell}(g=\ln F/G,h=\ln FG)=(FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_\ell}^{n_\ell}F\cdot G, \tag{6.3}$$

where$n_1+n_2+\cdots+n_\ell\geq 1$, and operators $D_{x_1},\cdots,D_{x_\ell}$ are classical Hirota's bilinear operators defined by

$$D_{x_1}^{n_1}\cdots D_{x_\ell}^{n_\ell}F{\cdot}G=(\partial_{x_1}-\partial_{x_1'})^{n_1}\cdots(\partial_{x_\ell}-\partial_{x_\ell'})^{n_\ell}F(x_1,\cdots,x_\ell)G(x_1',\cdots,x_\ell')|_{x_1'=x_1,\cdots,x_\ell'=x_\ell}.$$

In the special case of $F=G$, the formula (6.4) becomes

$$\begin{aligned}&F^{-2}D_{x_1}^{n_1}\cdots D_{x_\ell}^{n_\ell}G\cdot G=\mathcal{Y}_{n_1x_1,\cdots,n_\ell x_\ell}(0,q=2\ln G)\\&=\begin{cases}0, & n_1+\cdots+n_\ell \ \text{ is odd,}\\ P_{n_1x_1,\cdots,n_\ell x_\ell}(q), & n_1+\cdots+n_\ell \ \text{ is even.}\end{cases}\end{aligned} \tag{6.4}$$

The first few P-polynomial are

$$\begin{aligned}&P_{2x}(q)=q_{2x},\ P_{x,t}(q)=q_{xt},\ P_{4x}(q)=q_{4x}+3q_{2x}^2,\\&P_{6x}(q)=q_{6x}+15q_{2x}q_{4x}+15q_{2x}^3,\cdots.\end{aligned} \tag{6.5}$$

The formulas (6.4) and (6.5) will prove particularly useful in connecting nonlinear equations to their corresponding bilinear forms. This means that if a nonlinear equation is expressedby a linear combination of P-polynomials, then the nonlinear equation can be transformed into a linear equation.

Proposition 9. The binary Bell polynomials $\mathcal{Y}_{n_1x_1,\cdots,n_\ell x_\ell}(v,w)$ can be separated into P-polynomials and Y-polynomials

$$\begin{aligned}&(FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_\ell}^{n_\ell}F\cdot G=\mathcal{Y}_{n_1x_1,\cdots,n_\ell x_\ell}(g,h)|_{g=\ln F/G,h=\ln FG}\\&=\mathcal{Y}_{n_1x_1,\cdots,n_\ell x_\ell}(g,g+q,)|_{g=\ln F/G,q=2\ln G}\\&=\sum_{n_1+\cdots+n_\ell=even}\sum_{r_1=0}^{n_1}\cdots\sum_{r_\ell=0}^{n_\ell}\prod_{i=1}^{\ell}\binom{n_i}{r_i}P_{r_1x_1,\cdots,r_\ell x_\ell}(q)Y_{(n_1-r_1)x_1,\cdots,(n_\ell-r_\ell)x_\ell}(v).\end{aligned} \tag{6.6}$$

The key property of the multi-dimensional Bell polynomials

$$Y_{n_1x_1,\cdots,n_\ell x_\ell}(g)|_{g=\ln\psi}=\psi_{n_1x_1,\cdots,n_\ell x_\ell}/\psi, \tag{6.7}$$

implies that the binary Bell polynomials $\mathcal{Y}_{n_1x_1,\cdots,n_\ell x_\ell}(g,h)$ can still be linearized by means of the Hopf-Cole transformation $g=\ln\psi$, that is, $\psi=F/G$. The formulas (6.6) and (6.7) will then provide the shortest way to the associated Lax system of nonlinear equations.

6.2. Bilinearization

Theorem 6. Under the transformation

$$v = v_0 + 2(\ln G)_{2x}, \quad w = w_0 + 2(\ln G)_{xt},$$

the NKdV equation (1.1) can be bilinearized into

$$\begin{aligned} &(D_x^4 + 12v_0 D_x^2 - D_x D_y)G \cdot G = 0, \\ &(2D_t D_x^3 + 6w_0 D_x^2 + D_t D_y)G \cdot G = 0. \end{aligned} \tag{6.8}$$

where y ia an auxiliary variable, and u_0, v_0 are two constant solutions of the NKdV equation (1.1).

Proof. The invariance of the NKdV equation (1.1) under the scale transformation

$$x \to \lambda x, \quad t \to \lambda^\alpha t, \quad v \to \lambda^{-2} v, \quad w \to \lambda^{-\alpha-1} w$$

shows that the dimensions of the fields v and w are -2 and $-(\alpha+1)$, respectively. So we may introduce a dimensionless potential field q by setting

$$v = v_0 + q_{2x}, \quad w = w_0 - q_{xt}. \tag{6.9}$$

Substituting the transformation (6.9) into the equation (1.1), we can write the resulting equation in the following form

$$q_{4x,t} + 4q_{2x}q_{2x,t} + 2q_{3x}q_{xt} + 4v_0 q_{2x,t} + 2w_0 q_{3x} = 0,$$

which is regrouped as follows

$$\frac{2}{3}q_{4x,t} + 2(q_{2x}q_{2x,t} + q_{xt}q_{3x}) + \frac{1}{3}q_{4x,t} + 2q_{2x}q_{2x,t} + 4v_0 q_{2x,t} + 2w_0 q_{3x} = 0, \tag{6.10}$$

where we will see that Such an expression is necessary to get a bilinear form of the equation (1.1). Further integrating the equation (6.10) with respect to x yields

$$E(q) \equiv \frac{2}{3}(q_{3x,t} + 3q_{2x}q_{xt} + 3w_0 q_{2x}) + \frac{1}{3}\partial_x^{-1}\partial_t(q_{4x} + 3q_{2x}^2 + 12v_0 q_{2x}) = 0. \tag{6.11}$$

In order to write the equation (6.11) in a local bilinear form, let us first get rid of the integral operator ∂_x^{-1}. To do so, we introduce an auxiliary variable y and impose a subsidiary constraint condition

$$q_{4x} + 3q_{2x}^2 + 12v_0 q_{2x} - q_{xy} = 0. \tag{6.12}$$

Then, the equation (6.10) becomes

$$2(q_{3x,t} + 3q_{2x}q_{xt} + 3w_0 q_{2x}) + q_{yt} = 0. \tag{6.13}$$

According to the formula (6.5), the equations (6.12) and (6.13) are then cast into a pair of equations in the form of P-polynomials

$$P_{4x}(q) + 12v_0 P_{2x}(q) - P_{xy}(q) = 0,$$

$$2P_{3x,t}(q) + 6w_0 P_{2x}(q) + P_{yt}(q) + 3\gamma = 0.$$

Finally, by the property (6.4), making the following variable

$$q = 2\ln G \iff v = v_0 + 2(\ln G)_{2x}, \quad w = w_0 + 2(\ln G)_{xt},$$

change above system to the following bilinear forms of the NKdV equation (1.1) as follows

$$\begin{aligned} &(D_x^4 + 12v_0 D_x^2 - D_x D_y)G \cdot G = 0, \\ &(2D_t D_x^3 + 6w_0 D_x^2 + D_t D_y)G \cdot G = 0, \end{aligned} \tag{6.14}$$

which is also simultaneously bilinear system in y. This system is easily solved with multi-soliton solutions by using the Hirota's bilinear method. $\square$

Finally, we show that the NKdV-1 equation (1.1) can be directly bilinearized through a transformation, not Bell polynomials. Making dependent variable transformation

$$v = v_0 + 2(\ln F)_{xx}, \quad u = G/F, \tag{6.15}$$

we can change the equation (1.2) into

$$2(F_{xt} - F_x F_t) = G^2,$$

$$F_{xx}G - 2F_x G_x + G_{xx}F + v_0 FG = 0,$$

which is equivalent to the bilinear form

$$D_x D_t F \cdot F = G^2, \quad (D_x^2 + v_0)F \cdot G = 0. \tag{6.16}$$

It is obvious that the bilinear form of the NKdV-1 (6.16) is more simple than the bilinear form of NKdV (6.15).

6.3. N-soliton solutions

As usual as the normal perturbation method, let us expand G in the power series of a small parameter ε as follows

$$G = 1 + \varepsilon g^{(1)} + \varepsilon^2 g^{(2)} + \varepsilon^3 g^{(3)} + \cdots$$

Substituting the above equation into (6.7) and sorting each order of ε, we have

$$\begin{aligned} \varepsilon:\quad & (D_x^4 + 12v_0 D_x^2 - D_x D_y) g_1 \cdot 1 = 0, \\ & (2D_t D_x^3 + 6w_0 D_x^2 + D_t D_y) g^{(1)} \cdot 1 = 0, \end{aligned} \tag{6.16}$$

$$\begin{aligned} \varepsilon^2:\quad & (D_x^4 + 12v_0 D_x^2 - D_x D_y)(2g^{(2)} \cdot 1 + g^{(1)} \cdot g^{(1)}) = 0, \\ & (2D_t D_x^3 + 6w_0 D_x^2 + D_t D_y)(2g^{(2)} \cdot 1 + g^{(1)} \cdot g^{(1)}) = 0, \end{aligned} \tag{6.17}$$

$$\begin{aligned} \varepsilon^3:\quad & (D_x^4 + 12v_0 D_x^2 - D_x D_y)(g^{(3)} \cdot 1 + g^{(1)} \cdot g^{(2)}) = 0, \\ & (2D_t D_x^3 + 6w_0 D_x^2 + D_t D_y)(g^{(3)} \cdot 1 + g^{(1)} \cdot g^{(2)}) = 0, \end{aligned} \tag{6.18}$$

$$\cdots\cdots$$

By employing formulae mentioned above, the system (6.16) is equivalent to the following linear system

$$\begin{aligned} & g^{(1)}_{xxxx} + 12v_0 g^{(1)}_{xx} - g^{(1)}_{xy} = 0, \\ & 2g^{(1)}_{xxxt} + 6w_0 g^{(1)}_{xx} + g^{(1)}_{yt} = 0, \end{aligned}$$

which has solution

$$g^{(1)} = e^{\xi}, \quad \xi = kx - \frac{2kw_0}{k^2 + 4v_0} t + (k^3 + 12v_0 k)y + \sigma, \tag{6.19}$$

where k and σ are two arbitrary parameters.

Substituting (6.12) into (6.10) and (6.11) and choosing $g^{(2)} = g^{(3)} = \cdots = 0$, then the G's expansion is truncated with a finite sum as

$$G = 1 + e^{\xi},$$

which gives regular one-soliton solution of the NKdV equation (1.1)

$$\begin{aligned} & v = v_0 + 2\partial_x^2 \ln(1 + e^{\xi}) = v_0 + \frac{k^2}{2} \mathrm{sech}^2 \xi/2, \\ & w = w_0 + 2\partial_t \partial_x \ln(1 + e^{\xi}) = w_0 + \frac{k^2 w_0}{k^2 + 4v_0} \mathrm{sech}^2 \xi/2, \\ & \xi = kx - \frac{2kw_0}{k^2 + 4v_0} t + \gamma, \end{aligned} \tag{6.20}$$

where $\gamma = (k^3 + 12v_0 k)y + \sigma$, and k, v_0, w_0 are constants.

Let $w_0 = 1,\ v_0 = 0$, then the solution (6.20) reads as a kink-type solution of the NKdV-I equation (1.2)

$$u = \pm \tanh \xi/2, \quad \xi = kx - \frac{2}{k}t + \gamma.$$

In a similar way, taking

$$g^{(1)} = e^{\xi_1} + e^{\xi_2}, \ \ \xi_j = k_j x - \frac{2k_j w_0}{k_j^2 + 4v_0}t + \gamma_j, \ \ j = 1, 2,$$

we get a two-soliton wave solution

$$\begin{aligned} v &= v_0 + 2\partial_x^2 \ln(1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_1+\xi_2+A_{12}}) \\ w &= w_0 - 2\partial_t\partial_x \ln(1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_1+\xi_2+A_{12}}), \\ A_{12} &= \ln\left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2. \end{aligned} \tag{6.21}$$

In general, we can get a N-soliton solution of the NKdV equation (1.1)

$$\begin{aligned} v &= v_0 + 2\partial_x^2 \ln\left(\sum_{\mu_j=0,1} \exp(\sum_{j=1}^{N} \mu_j\xi_j + \sum_{1\le j\le N}^{N} \mu_j\mu_l A_{jl}\right), \\ w &= w_0 - \partial_t\partial_x \ln\left(\sum_{\mu_j=0,1} \exp(\sum_{j=1}^{N} \mu_j\xi_j + \sum_{1\le j\le N}^{N} \mu_j\mu_l A_{jl}\right), \\ A_{jl} &= \ln\left(\frac{k_j - k_l}{k_j + k_l}\right)^2. \end{aligned}$$

where the notation $\sum_{\mu_j=0,1}$ represents all possible combinations $\mu_j = 0, 1$, and $\xi_j = k_j x - \frac{2k_j w_0}{k_j^2+4v_0}t + \gamma_j, \ \ j = 1, 2, \cdots, N.$

In the following, we discuss the soliton solutions for NKdV-1 equation by using bilinear equation (6.16). Let us expand F and G in the power series of a small parameter ε as follows

$$F = 1 + f^{(2)}\varepsilon^2 + f^{(4)}\varepsilon^4 + f^{(6)}\varepsilon^6 + \cdots$$

$$G = g^{(1)}\varepsilon + g^{(3)}\varepsilon^3 + g^{(5)}\varepsilon^5 + \cdots$$

Substituting the above equation into (6.16) and arranging each order of ε, we have

$$\begin{aligned}
&g^{(1)}_{xx} + v_0 g^{(1)} = 0,\\
&g^{(3)}_{xx} + v_0 g^{(3)} = -(D_x^2 + v_0) f^{(2)} \cdot g^{(1)},\\
&g^{(5)}_{xx} + v_0 g^{(5)} = -(D_x^2 + v_0)(f^{(2)} \cdot g^{(3)} + f^{(4)} \cdot g^{(1)}),\\
&\cdots\cdots
\end{aligned} \tag{6.23}$$

$$\begin{aligned}
&2f^{(2)}_{xt} = (g^{(1)})^2,\\
&2f^{(4)}_{xt} = 2g^{(1)}g^{(3)} - D_x D_t f^{(2)} \cdot f^{(2)},\\
&2f^{(6)}_{xt} = 2g^{(1)}g^{(5)} + 2(g^{(3)})^2 - 2D_x D_t f^{(3)} \cdot f^{(3)},\\
&\cdots\cdots
\end{aligned} \tag{6.24}$$

Let $v_0 = -k^2$, it follows from the first equation of (6.23) and (6.24) that

$$g^{(1)} = e^{\xi}, \quad f^{(2)} = \frac{1}{4}e^{2\xi}, \quad \xi = kx + \frac{1}{2k}t + \gamma. \tag{6.25}$$

Substituting (6.25) into the second equation of (6.23) leads to

$$g^{(3)}_{xx} - k^2 g^{(3)} = 0,$$

from which we may take $g^{(3)} = 0$, further choose $g^{(5)} = \cdots = 0,\ f^{(4)} = \cdots = 0$. So F and G are truncated with a finite sum as

$$F = 1 + \frac{1}{4}e^{2\xi}, \quad G = e^{\xi}.$$

Finally, the formula (6.14) gives one-soliton solution of the NKdV-1 equation (1.2)

$$v = 2k^2 \mathrm{sech}^2\xi - k^2, \quad u = \mathrm{sech}\xi.$$

7. Bilinear Bäcklund transformation

In this section, we search for the bilinear Bäcklund transformation and Lax pair of the NKdV equation (1.1).

7.1. Bilinear Bäcklund transformation

Theorem 7. Suppose that F is a solution of the bilinear equation (6.8), and if G satisfying

$$\begin{aligned}&(D_x^2-\lambda)F\cdot G=0,\\&[D_tD_x^2+2w_0D_x+(4v_0+3\lambda)D_t]F\cdot G=0,\end{aligned}\tag{7.1}$$

then G is another solution of the equation (6.8).

Proof. Let

$$q=2\ln G,\quad \tilde{q}=2\ln F$$

be two different solutions of the equation (6.10). Introducing two new variables

$$h=(\tilde{q}+q)/2=\ln(FG),\quad g=(\tilde{q}-q)/2=\ln(F/G),$$

makes the function E invariant under the two fields $\tilde{q}$ and q:

$$\begin{aligned}&E(\tilde{q})-E(q)=E(h+g)-E(h-g)\\&=8v_0g_{xt}+4w_0g_{2x}+2g_{3x,t}+4h_{2x}g_{x,t}+4h_{x,t}g_{2x}+4\partial_x^{-1}(h_{2x}g_{2x,t}+h_{2x,t}g_{2x})\\&=2\partial_x(\mathcal{Y}_{2x,t}(g,h)+4v_0\mathcal{Y}_t(g)-2w_0\mathcal{Y}_x(g))+R(g,h)=0,\end{aligned}\tag{7.2}$$

where

$$R(g,h)=-2\partial_x[(h_{2x}+g_x^2)g_t]+4h_{2x}g_{xt}-4h_{2x,t}g_x+4\partial_x^{-1}(h_{2x}g_{2x,t}+h_{2x,t}g_{2x}).$$

This two-field invariant condition can be regarded as a natural ansatz for a bilinear Bäcklund transformation and may produce some required transformations under additional appropriate constraints.

In order to decouple the two-field condition (7.2),let us impose a constraint so as to express $R(g,h)$ in the form of x-derivative of $\mathcal{Y}$-polynomials. The simple choice of the constraint may be

$$\mathcal{Y}_{2x}(g,h)=h_{2x}+g_x^2=\lambda,\tag{7.3}$$

which directly leads to

$$R(g,h)=2\lambda g_{xt}+4h_{2x}g_{xt}-4h_{2x,t}g_x-4g_x^2g_{xt}=6\lambda g_{xt},\tag{7.4}$$

where $h_{2x,t}=-2g_xg_{xt}$ and $h_{2x}=\lambda-g_x^2$ are used.

Using the relations (7.2)-(7.4), we derived a coupled system of $\mathcal{Y}$-polynomials

$$\begin{aligned}&\mathcal{Y}_{2x}(g,h)-\lambda=0,\\&\mathcal{Y}_{2x,t}(g,h)+(4v_0+3\lambda)\mathcal{Y}_t(g)+2w_0\mathcal{Y}_x(g)=0,\end{aligned}\tag{7.5}$$

where we prefer the second equation to be expressed in the form of conserved quantity without integration with respect to x. This is very useful to construct conservation laws. Apparently, the identity (6.2) directly sends the system (7.5) to the following bilinear Bäcklund transformation

$$\begin{aligned}&(D_x^2-\lambda)F\cdot G=0,\\&[D_tD_x^2+2w_0D_x+(4v_0+3\lambda)D_t]F\cdot G=0,\end{aligned}\tag{7.6}$$

where we have integrated the second equation in the system (7.5) with respect to x, and w_0 is the corresponding integration constant. □

7.2. Inverse scattering formulation

Theorem 8. The NKdV equation (1.1) admits a Lax pair

$$\begin{aligned}&\psi_{2x}+v\psi=\lambda\psi,\\&4\psi_{2x,t}+4v\psi_t-2w\psi_x-3w_x\psi=0.\end{aligned}\tag{7.7}$$

Proof. By the transformation $v=\ln\psi$, it follows from the formulas (6.5) and (6.6) that

$$\begin{aligned}&\mathcal{Y}_t(g)=\psi_t/\psi,\quad \mathcal{Y}_x(g)=\psi_x/\psi,\quad \mathcal{Y}_{2x}(g,h)=q_{2x}+\psi_{2x}/\psi,\\&\mathcal{Y}_{2x,t}(g,h)=2q_{xt}\psi_x/\psi+q_{2x}\psi_t/\psi+\psi_{2x,t}/\psi,\end{aligned}$$

which make the system (7.5) linearized into a Lax pair with parameter λ

$$L\psi\equiv(\partial_x^2+q_{2x})\psi=\lambda\psi,\tag{7.8}$$

$$M\psi\equiv[\partial_t\partial_x^2+(4v_0+q_{2x})\partial_t+2(q_{xt}+w_0)\partial_x+3\lambda\partial_t]\psi,\tag{7.9}$$

or equivalently,

$$\begin{aligned}&\psi_{2x}+v\psi=\lambda\psi,\\&4\psi_{2x,t}+4v\psi_t-2w\psi_x-3w_x\psi=0,\end{aligned}$$

where the equation (7.8) is used to get the second equation. One can easily verify from equations (7.8) and (7.9) that

$$[L,M]=q_{4x,t}+4(v_0+q_{2x})q_{2x,t}+2q_{3x}(q_{xt}+w_0)=0$$

exactly gives the NKdV equation (1.1) through replacing $v_0 + q_{2x}$ and $w_0 + q_{xt}$ by v and w, respectively. $\square$

8. Darboux covariant Lax pair

In this section, we will give a kind of Darboux covariant Lax pair, whose form is invariant under the gauge transformation (4.3).

Theorem 9. The NKdV equation (1.1) possesses the following Darboux covariant Lax pair

$$
\begin{aligned}
&L\psi = \lambda\psi, \\
&M_{\mathrm{cov}}\psi = 0, \quad M_{\mathrm{cov}} = M + 3\partial_x L,
\end{aligned}
$$

under the gauge transformation $\tilde{\psi} = T\psi$. This is actually equivalent to the Lax pair (2.9).

Proof. In section 4, we have shown that the gauge transformation (4.1) maps the operator $L(q)$ onto a similar operator

$$\tilde{L}(\tilde{q}) = TL(q)T^{-1},$$

which satisfies the following covariance condition

$$\tilde{L}(\tilde{q}) = L(q + \Delta q), \;\; \tilde{q} = q + \Delta q, \;\text{ with }\;\; \Delta q = 2\ln\phi.$$

Next, we want to find a third order operator $M_{\mathrm{cov}}(q)$ with appropriate coefficients, such that $M_{\mathrm{cov}}(q)$ is mapped by gauge transformation (8.1) onto a similar operator $\tilde{M}_{\mathrm{cov}}(\tilde{q})$, which satisfies the covariance condition

$$\tilde{M}_{\mathrm{cov}}(\tilde{q}) = M_{\mathrm{cov}}(q + \Delta q), \;\; \tilde{q} = q + \Delta q.$$

Suppose that ϕ is a solution of the following Lax pair

$$
\begin{aligned}
&L\psi = \lambda\psi, \\
&M_{\mathrm{cov}}\psi = 0, \quad M_{\mathrm{cov}} = 4\partial_t\partial_x^2 + b_1\partial_x + b_2\partial_t + b_3,
\end{aligned}
\tag{8.2}
$$

where b_1, b_2 and b_3 are functions to be determined. Then, the transformation T is required to map the operator M_{cov} to the similar one

$$TM_{\mathrm{cov}}T^{-1} = \tilde{M}_{\mathrm{cov}}, \quad \tilde{L}_{2,\mathrm{cov}} = 4\partial_t\partial_x^2 + \tilde{b}_1\partial_x + \tilde{b}_2\partial_t + \tilde{b}_3, \tag{8.3}$$

where $\tilde{b}_1, \tilde{b}_2$ and $\tilde{b}_3$ satisfy the covariant condition

$$\tilde{b}_j = b_j(q) + \Delta b_j = b_j(q + \Delta q), \quad j = 1, 2, 3. \tag{8.4}$$

It follows from (8.2) and (5.3) that

$$\Delta b_1 = \tilde{b}_1 - b_1 = 4\sigma_t, \quad \Delta b_2 = \tilde{b}_2 - b_2 = 8\sigma_x, \tag{8.5}$$

$$\Delta b_3 = \tilde{b}_3 - b_3 = \sigma \Delta b_1 + 8\sigma_{xt} + b_{1,x}, \tag{8.6}$$

and σ satisfy

$$4\sigma_{2x,t} + \tilde{b}_1 \sigma_x + \tilde{b}_2 \sigma_t + \sigma \Delta b_3 + b_{3,x} = 0. \tag{8.7}$$

According to the relation (8.4), it remains to determine $b_1,\ b_2$ and b_3 in the form of polynomial expressions in terms of q's derivatives

$$b_j = F_j(q, q_x, q_y, q_{xy}, q_{2x}, q_{2y}, q_{2x,y}, \cdots), \quad j = 1, 2, 3$$

such that

$$\Delta F_j = F_j(q + \Delta q, q_x + \Delta q_x, q_t + \Delta q_t, \cdots) - F_j(q, q_x, q_t, \cdots) = \Delta b_j, \tag{8.8}$$

with $\Delta q_{kx,lt} = 2(\ln \phi)_{kx,lt},\ k, l = 1, 2, \cdots$, and Δb_j being given through the relations (8.5)-(8.7).

Expanding the left hand of the equation (8.8), we obtain

$$\Delta b_1 = \Delta F_1 = F_{1,q}\Delta q + F_{1,q_x}\Delta q_x + F_{1,q_y}\Delta q_y + F_{1,q_{xy}}\Delta q_{xt} + \cdots = 4\sigma_t = 2\Delta q_{xt},$$

which implies that we can determine b_1 up to a arbitrary constant c_1, namely,

$$b_1 = F_1(q_{xt}) = 2q_{xt} + c_1,\ c_1 \text{ is an arbitrary constant} \tag{8.9}$$

Proceeding in the same way deduce the function b_2 as follows

$$b_2 = F_2(q_{2x}) = 4q_{2x} + c_2, \tag{8.10}$$

where c_2 is an arbitrary constant.

We see from the relation (8.6) that Δb_3 contains the term $b_{1,x} = q_{2x,t}$, which should be eliminated such that Δb_3 admits the form (8.8). By the Lax pair (8.2), we have the following relation

$$q_{2x,t} = -\sigma_{xt} - 2\sigma\sigma_t. \tag{8.11}$$

Substituting (8.9) and (8.11) into (8.6) yields

$$\Delta b_3 = 4\sigma\sigma_t + 8\sigma_{xt} + 2q_{2x,t} = 6\sigma_{xt} = 3\Delta q_{2x,t}.$$

If choosing

$$b_3 = F_3(q_{2x,t}) = 3q_{2x,t} + c_3, \tag{8.12}$$

the third condition

$$\Delta F_3 = F_{3,q}\Delta q + F_{3,q_x}\Delta q_x + F_{3,q_t}\Delta q_t \cdots = \Delta b_3$$

holds, where c_3 is an arbitrary constant.

Letting $c_1 = -2v_0,\ c_2 = 0,\ c_3 = w_0$ in (8.9), (8.10) and (8.12), then it follows from (8.2) that we have the following Darboux covariant evolution equation

$$M_{\rm cov}\psi = 0, \quad M_{\rm cov} = 4\partial_t\partial_x^2 + 2q_{xt}\partial_x + 4q_{2x}\partial_t + 3q_{2x,t},$$

which coincides with the equation (8.7). Moreover, the relation between two operators $L_{2,\rm cov}$ and L_2 are related through

$$M_{\rm cov} = M + 3\partial_x L.$$

The compatibility condition of the Darboux covariant Lax pair (8.2) exactly gives the NKdV equation(1.1) in Lax representation

$$\begin{aligned}[M_{\rm cov}, L] &= q_{4x,t} + 4(v_0 + q_{2x})q_{2x,t} + q_{3x}(q_{xt} + w_0)\\ &= v_{xxx} + 4vw_x + 2v_xw = 0.\end{aligned}$$

□

In the above repeated procedure, we are able to obtain higher order operators, which are also Darboux covariant with respect to T, to produce higher order members of the negative order KdV hierarchy.

9. Conservation laws of NKdV equations

In this section, we will present infinitely many conservation laws in a local form for the NKdV equation (1.1) based on a generalized Miura transformation.

Theorem 10. The NKdV equation (1.1) possesses the following infinitely many conservation laws

$$F_{n,t} + G_{n,x} = 0,\ \ n = 1, 2, \cdots . \tag{9.1}$$

where the conversed densities $F_n's$ are recursively given by recursion formulas explicitly

$$\begin{aligned} &F_0 = v_{xx} - v^2, \quad F_1 = -v_{xxx} + 2vv_x, \\ &F_n = I_{n,xx} - \sum_{k=0}^{n} I_k I_{n-k} + \sum_{k=0}^{n-2} I_k I_{n-2-k,x}, \quad n = 2, 3, \cdots . \end{aligned} \tag{9.2}$$

and the fluxes $G_n's$ are

$$\begin{aligned} &G_0 = 2wI_0 = 2wv, \quad G_1 = 2wI_1 = -2wv_x, \\ &G_n = 2wI_n, \quad n = 2, 3, \cdots . \end{aligned} \tag{9.3}$$

Proof. For the simplicity, let us select $v_0 = w_0 = 0$ in the transformation (6.9). We introduce a new potential function

$$q_{2x} = \eta + \varepsilon\eta_x + \varepsilon^2\eta^2, \tag{9.4}$$

where ε is a constant parameter. Substituting (9.4) into the Lax equation (7.10) leads to

$$0 = [L, M] = (1 + \varepsilon\partial_x + 2\varepsilon^2\eta)[-4(\eta + \varepsilon^2\eta^2)\eta_t - 2(q_x - \varepsilon\eta)_t\eta_x + \eta_{2x,t}],$$

which implies that $v = q_{2x},\ w = q_{xt}$ given by (9.4) are a solution of the NKdV equation (1.1) if η satisfies the following equation

$$-4(\eta + \varepsilon^2\eta^2)\eta_t - 2(q_x - \varepsilon\eta)_t\eta_x + \eta_{2x,t} - 4\eta_t = 0. \tag{9.5}$$

On the other hand, it follows from (9.5) that

$$[(q_x - \varepsilon\eta)_t]_x = -(\eta + \varepsilon^2\eta^2)_t.$$

Therefore, the equation (9.5) can be rewritten as

$$(\eta_{2x} - \eta^2)_t + [2\eta(\varepsilon^2\eta - q_x)_t]_x = 0,$$

or a divergent-type form

$$(\eta_{2x} + 2\varepsilon^2\eta\eta_x - \eta^2)_t + (2\eta w)_x = 0 \tag{9.6}$$

by replacing $q_{xt} = w$.

Inserting the expansion

$$\eta = \sum_{n=0}^{\infty} I_n(q, q_x, q_t \cdots)\varepsilon^n, \tag{9.7}$$

into the equation (9.4) and comparing the coefficients for power of ε, we obtain the recursion relations to calculate I_n in an explicit form

$$\begin{aligned} &I_0 = q_{2x} = v, \quad I_1 = -I_{0,x} = -v_x, \\ &I_n = -I_{n-1,x} - \sum_{k=0}^{n-2} I_k I_{n-2-k}, \quad n = 2, 3, \cdots . \end{aligned} \tag{9.8}$$

Substituting (9.7) into (9.6) and simplifying terms in the power of ε provide us infinitely many conservation laws

$$F_{n,t} + G_{n,x} = 0, \ \ n = 1, 2, \cdots$$

where the conversed densities $F_n' s$ and the fluxes $G_n' s$ are by (9.2) and (9.3), respectively. □

Here, we already give recursion formulas (9.7) and (9.8) to show how to generate conservation laws (9.6) based on the first few explicitly provided. Apparently, the first equation in conservation laws (9.6)

$$v_{xxt} - 2vv_t + 2wv_x + 2w_x v = 0$$

is exactly the NKdV equation (1.1)

$$\begin{aligned} &v_t + w_x = 0, \\ &w_{xxx} + 4vw_x + 2wv_x = 0. \end{aligned}$$

which is reduced to the NKdV equation (1.2) under the constraints $v = -u_{xx}/u$ and $w = u^2$.

In conclusion, the NKdV equation (1.1) is completely integrable and admits bilinear Bäcklund transformation, Lax pair and infinitely many local conservation laws.

10. Quasi-periodic solutions of the NKdV equation

In this section, we study quasi-periodic wave solutions of the NKdV equation (1.1) by using bilinear Bäcklund transformation (7.1) and bilinear formulas derived in section 9.

In fact, a quasi-periodic solution, also called algebro-geometric solutions or finite gap solutions, was originally studied in the late 1970s by Novikov, Dubrovin, McKean, Lax, Its, and Matveev et al [11, 36, 49, 57], based on the inverse spectral theory and algebro-geometric method. In recent years, this theory has been extended to a large class of nonlinear integrable equations including sine-Gordon equation, Camassa-Holm equation, Thirring model equation, Kadomtsev-Petviashvili equation, Ablowitz-Ladik lattice, and Toda lattice [7, 21, 22, 23, 25, 24, 31, 67, 62, 68, 82, 87, 86]. The algebro-geometric theory, however, needs Lax pairs and is also involved in complicated analysis procedure on the Riemann surfaces. It is rather difficult to directly determine the characteristic parameters of waves, such as frequencies and phase shifts for a function with given wave-numbers and amplitudes. On the other hand, the bilinear derivative method developed by Hirota is a powerful approach for constructing exact solution of nonlinear equations in an explicit form. If a nonlinear equation is able to be written in a bilinear form by a dependent variable transformation, then multi-solitary wave solutions are usually obtained for the equation [29, 30, 34, 35, 84]. Based on the Hirota forms, Nakamura proposed a convenient way to find a kind of explicit quasi-periodic solutions of nonlinear equations [56], where the periodic wave solutions of the KdV equation and the Boussinesq equation were obtained. Such a method indeed displays some advantages over algebro-geometric methods. For example, it does not need any Lax pair and Riemann surface for the given nonlinear equation, and is also able to find the explicit construction of multi-periodic wave solutions. The method relies on the existence of the Hirota's bilinear form as well as arbitrary parameters appearing in Riemann matrix [13, 14].

10.1. Multi-dimensional Riemann theta functions

Let us first begin with some preliminary work about multi-dimensional Riemann theta functions and their quasi-periodicity. The multi-dimensional Riemann theta function is defined by

$$\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon},\boldsymbol{s}|\boldsymbol{\tau})=\sum_{\boldsymbol{n}\in\mathbb{Z}^N}\exp\{2\pi i\langle\boldsymbol{\zeta}+\boldsymbol{\varepsilon},\boldsymbol{n}+\boldsymbol{s}\rangle-\pi\langle\boldsymbol{\tau}(\boldsymbol{n}+\boldsymbol{s}),\boldsymbol{n}+\boldsymbol{s}\rangle\},\tag{10.1}$$

where $\boldsymbol{n}=(n_1,\cdots,n_N)^T\in\mathbb{Z}^N$ is an integer value vector, and $\boldsymbol{s}=(s_1,\cdots,s_N)^T,\boldsymbol{\varepsilon}=(\varepsilon_1,\cdots,\varepsilon_N)^T\in\mathbb{C}^N$ is a complex parameter vector. $\boldsymbol{\zeta}=(\zeta_1,\cdots,\zeta_N)^T,\ \zeta_j=\alpha_jx+\beta_jt+\delta_j,\ \ \alpha_j,\beta_j,\delta_j\in\Lambda_0,\ j=1,2,\cdots,N$ are complex phase variables, where x,t are ordinary real variables and θ is a Grassmann variable. The inner product of two vectors $\boldsymbol{f}=(f_1,\cdots,f_N)^T$ and $\boldsymbol{g}=(g_1,\cdots,g_N)^T$ is defined by

$$\langle\boldsymbol{f},\boldsymbol{g}\rangle=f_1g_1+f_2g_2+\cdots+f_Ng_N.$$

The matrix $\boldsymbol{\tau}=(\tau_{ij})$ is a positive definite and real-valued symmetric $N\times N$ matrix. The entries τ_{ij} of the periodic matrix $\boldsymbol{\tau}$ can be considered as free parameters of the theta function (10.1).

In this paper, we choose τ to be purely imaginary matrix to make the theta function (10.1) real-valued. In definition (10.1) for the case of $\boldsymbol{s}=\boldsymbol{\varepsilon}=\mathbf{0}$, we denote $\vartheta(\boldsymbol{\zeta},\boldsymbol{\tau})=\vartheta(\boldsymbol{\zeta},\mathbf{0},\mathbf{0}|\boldsymbol{\tau})$ for simplicity. Therefore, we have $\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon},\mathbf{0}|\boldsymbol{\tau})=\vartheta(\boldsymbol{\zeta}+\boldsymbol{\varepsilon},\boldsymbol{\tau})$.

Remark 4. The above periodic matrix $\boldsymbol{\tau}$ is different from the one in the algebro-geometric approach discussed in [57]-[14], where it is usually constructed on a compact Riemann surface Γ with genus $N\in\mathbb{N}$. One may see that the entries in the matrix $\boldsymbol{\tau}$ are not free and difficult to be explicitly given. $\square$

Definition 3. A function $g(\boldsymbol{x},t)$ on $\mathbb{C}^N\times\mathbb{C}$ is said to be quasi-periodic in t with fundamental periods $T_1,\cdots,T_k\in\mathbb{C}$ if $T_1,\cdots,T_k$ are linearly dependent over $\mathbb{Z}$ and there exists a function $G(\boldsymbol{x},\boldsymbol{y})\in\mathbb{C}^N\times\mathbb{C}^k$ such that

$$G(\boldsymbol{x},y_1,\cdots,y_j+T_j,\cdots,y_k)=G(\boldsymbol{x},y_1,\cdots,y_j,\cdots,y_k),\ \text{ for all}\, y_j\in\mathbb{C},\ j=1,\cdots,k.$$

$$G(\boldsymbol{x},t,\cdots,t,\cdots,t)=g(\boldsymbol{x},t).$$

In particular, $g(\boldsymbol{x},t)$ becomes periodic with T if and only if $T_j=m_jT$. $\square$

Let's first see periodicity of the theta function $\vartheta(\boldsymbol{\zeta},\boldsymbol{\tau})$.

Proposition 10. [55] Let $\boldsymbol{e_j}$ be the $j-$th column of $N \times N$ identity matrix I_N; τ_j be the $j-$th column of $\boldsymbol{\tau}$, and τ_{jj} the (j,j)-entry of $\boldsymbol{\tau}$. Then the theta function $\vartheta(\boldsymbol{\zeta},\boldsymbol{\tau})$ has the periodic properties

$$\vartheta(\boldsymbol{\zeta}+\boldsymbol{e_j}+i\boldsymbol{\tau_j},\boldsymbol{\tau}) = \exp(-2\pi i\zeta_j+\pi\tau_{jj})\vartheta(\boldsymbol{\zeta},\boldsymbol{\tau}).$$

The theta function $\vartheta(\boldsymbol{\zeta},\boldsymbol{\tau})$ which satisfies the condition (5.4) is called a multiplicative function. We regard the vectors $\{\boldsymbol{e_j}, \quad j=1,\cdots,N\}$ and $\{i\boldsymbol{\tau_j}, \quad j=1,\cdots,N\}$ as periods of the theta function $\vartheta(\boldsymbol{\zeta},\boldsymbol{\tau})$ with multipliers 1 and $\exp(-2\pi i\zeta_j+\pi\tau_{jj})$, respectively. Here, only the first N vectors are actually periods of the theta function $\vartheta(\boldsymbol{\zeta},\boldsymbol{\tau})$, but the last N vectors are the periods of the functions $\partial^2_{\zeta_k,\zeta_l}\ln\vartheta(\boldsymbol{\zeta},\boldsymbol{\tau})$ and $\partial_{\zeta_k}\ln[\vartheta(\boldsymbol{\zeta}+\boldsymbol{e},\boldsymbol{\tau})/\vartheta(\boldsymbol{\zeta}+\boldsymbol{h},\boldsymbol{\tau})]$, $k,l=1,\cdots,N$.

Proposition 11. Let $\boldsymbol{e_j}$ and $\boldsymbol{\tau_j}$ be defined as above proposition 2. The meromorphic functions $f(\boldsymbol{\zeta})$ are as follow

$$(i) \qquad f(\boldsymbol{\zeta}) = \partial^2_{\zeta_k\zeta_l}\ln\vartheta(\boldsymbol{\zeta},\boldsymbol{\tau}), \quad \boldsymbol{\zeta}\in C^N, \quad k,l=1,\cdots,N,$$

$$(ii) \qquad f(\boldsymbol{\zeta}) = \partial_{\zeta_k}\ln\frac{\vartheta(\boldsymbol{\zeta}+\boldsymbol{e},\boldsymbol{\tau})}{\vartheta(\boldsymbol{\zeta}+\boldsymbol{h},\boldsymbol{\tau})}, \quad \boldsymbol{\zeta},\ \boldsymbol{e},\ \boldsymbol{h}\in C^N, \quad j=1,\cdots,N.$$

then in all two cases (i) and (ii), it holds that

$$f(\boldsymbol{\zeta}+\boldsymbol{e_j}+i\boldsymbol{\tau_j}) = f(\boldsymbol{\zeta}), \quad \boldsymbol{\zeta}\in C^N, \quad j=1,\cdots,N,$$

which implies that $f(\boldsymbol{\zeta})$ is a quasi-periodic function.

10.2. Bilinear formulae of theta functions

To construct a kind of explicitly quasi-periodic solutions of the NKdV equation (1.1), we propose some important bilinear formulas of multi-dimensional Riemann theta functions, whose derivations are similar to the case of super bilinear equations [15], so we just list them without proofs.

Theorem 11. Suppose that $\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon'},\boldsymbol{0}|\boldsymbol{\tau})$ and $\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon},\boldsymbol{0}|\boldsymbol{\tau})$ are two Riemann theta functions, in which $\boldsymbol{\varepsilon}=(\varepsilon_1,\ldots,\varepsilon_N)$, $\boldsymbol{\varepsilon'}=(\varepsilon'_1,\ldots,\varepsilon'_N)$, and $\boldsymbol{\zeta}=(\zeta_1,\cdots,\zeta_N)$, $\zeta_j=\alpha_jx+\omega_jt+\delta_j, \quad j=1,2,\cdots,N$. Then operators D_x, D_t and S exhibit the

following perfect properties when they act on a pair of theta functions

$$\begin{aligned} &D_x\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon'},\mathbf{0}|\boldsymbol{\tau})\cdot\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon},\mathbf{0}|\boldsymbol{\tau})\\ &=\sum_{\boldsymbol{\mu}}\partial_x\vartheta(2\boldsymbol{\zeta},\boldsymbol{\varepsilon'}-\boldsymbol{\varepsilon},-\boldsymbol{\mu}/2|2\boldsymbol{\tau})|_{\boldsymbol{\zeta}=\mathbf{0}}\vartheta(2\boldsymbol{\zeta},\boldsymbol{\varepsilon'}+\boldsymbol{\varepsilon},\boldsymbol{\mu}/2|2\boldsymbol{\tau}), \end{aligned} \tag{10.2}$$

where $\boldsymbol{\mu}=(\mu_1,\cdots,\mu_N)$, and the notation $\sum_{\boldsymbol{\mu}}$ represents 2^N different transformations corresponding to all possible combinations $\mu_1=0,1;\cdots;\mu_N=0,1$.

In general, for a polynomial operator $H(D_x,D_t)$ with respect to D_x and D_t, we have the following useful formula

$$H(D_x,D_t)\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon'},\mathbf{0}|\boldsymbol{\tau})\cdot\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon},\mathbf{0}|\boldsymbol{\tau})=\sum_{\boldsymbol{\mu}}C(\boldsymbol{\varepsilon'},\boldsymbol{\varepsilon},\boldsymbol{\mu})\vartheta(2\boldsymbol{\zeta},\boldsymbol{\varepsilon'}+\boldsymbol{\varepsilon},\boldsymbol{\mu}/2|2\boldsymbol{\tau}), \tag{10.3}$$

in which, explicitly

$$C(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon'},\boldsymbol{\mu})=\sum_{\boldsymbol{n}\in\mathbb{Z}^N}H(\boldsymbol{\mathcal{M}})\exp\left[-2\pi\langle\boldsymbol{\tau}(\boldsymbol{n}-\boldsymbol{\mu}/2),\boldsymbol{n}-\boldsymbol{\mu}/2\rangle-2\pi i\langle\boldsymbol{n}-\boldsymbol{\mu}/2,\boldsymbol{\varepsilon'}-\boldsymbol{\varepsilon})\right]. \tag{10.4}$$

where we denote $\boldsymbol{\mathcal{M}}=(4\pi i\langle\boldsymbol{n}-\boldsymbol{\mu}/2,\boldsymbol{\alpha}\rangle,\ 4\pi i\langle\boldsymbol{n}-\boldsymbol{\mu}/2,\boldsymbol{\omega}\rangle)$.

Remark 6. The formulae (10.3) and (10.4) show that if the following equations are satisfied

$$C(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon'},\boldsymbol{\mu})=0, \tag{10.5}$$

for all possible combinations $\mu_1=0,1;\mu_2=0,1;\cdots;\mu_N=0,1$, in other word, all such combinations are solutions of equation (10.5), then $\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon'},\mathbf{0}|\boldsymbol{\tau})$ and $\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon},\mathbf{0}|\boldsymbol{\tau})$ are N-periodic wave solutions of the bilinear equation

$$H(D_x,D_t)\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon'},\mathbf{0}|\boldsymbol{\tau})\cdot\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon},\mathbf{0}|\boldsymbol{\tau})=0.$$

We call the formula (10.5) constraint equations, whose number is 2^N. This formula actually provides us an unified approach to construct multi-periodic wave solutions for supersymmetric equations. Once a supersymmetric equation is written bilinear forms, then its multi-periodic wave solutions can be directly obtained by solving system (10.5).

Theorem 12. Let $C(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon'},\boldsymbol{\mu})$ and $H(D_x,D_t)$ be given in Theorem 10, and make a choice such that $\varepsilon'_j-\varepsilon_j=\pm1/2,\ j=1,\cdots,N$. Then

(i) If $H(D_x, D_t)$ is an symmetric operator, i. e.

$$H(-D_x, -D_t) = H(D_x, D_t),$$

then $C(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu})$ vanishes automatically for the case when $\sum_{j=1}^{N} \mu_j$ is an odd number, namely

$$C(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu})|_{\boldsymbol{\mu}} = 0, \quad \text{for} \quad \sum_{j=1}^{N} \mu_j = 1, \text{ mod } 2.$$

(ii) If $H(D_x, D_t)$ is a skew-symmetric operator, i.e.

$$H(-D_x, -D_t) = -H(D_x, D_t),$$

then $C(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu})$ vanishes automatically for the case when $\sum_{j=1}^{N} \mu_j$ is an even number, namely

$$C(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu})|_{\boldsymbol{\mu}} = 0, \quad \text{for} \sum_{j=1}^{N} \mu_j = 0, \text{ mod } 2.$$

Proposition 12. Let $\varepsilon'_j - \varepsilon_j = \pm 1/2, \ j = 1, \cdots, N$. Assume $H(D_x, D_t)$ is a linear combination of even and odd functions

$$H(D_x, D_t) = H_1(D_x, D_t) + H_2(D_x, D_t),$$

where H_1 is even and H_2 is odd. In addition, $C(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu})$ corresponding (10.8) is given by

$$C(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu}) = C_1(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu}) + C_2(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu}),$$

where

$$C_1(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^N} H_1(\boldsymbol{\mathcal{M}}) \exp\left[-2\pi \langle \boldsymbol{\tau}(\boldsymbol{n} - \boldsymbol{\mu}/2), \boldsymbol{n} - \boldsymbol{\mu}/2 \rangle - 2\pi i \langle \boldsymbol{n} - \boldsymbol{\mu}/2, \boldsymbol{\varepsilon'} - \boldsymbol{\varepsilon} \rangle\right],$$

$$C_2(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^N} H_2(\boldsymbol{\mathcal{M}}) \exp\left[-2\pi \langle \boldsymbol{\tau}(\boldsymbol{n} - \boldsymbol{\mu}/2), \boldsymbol{n} - \boldsymbol{\mu}/2 \rangle - 2\pi i \langle \boldsymbol{n} - \boldsymbol{\mu}/2, \boldsymbol{\varepsilon'} - \boldsymbol{\varepsilon} \rangle\right].$$

Then

$$C(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu}) = C_2(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu}) \quad \text{for} \quad \sum_{j=1}^{N} \mu_j = 1, \text{ mod } 2,$$

$$C(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu}) = C_1(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu}), \quad \text{for} \sum_{j=1}^{N} \mu_j = 0, \text{ mod } 2.$$

The theorem 2 and corollary 1 are very useful to deal with coupled super-Hirota's bilinear equations, which will be seen in the following section 10.

By introducing differential operators

$$\nabla = (\partial_{\zeta_1}, \partial_{\zeta_2}, \cdots, \partial_{\zeta_N}),$$
$$\partial_x = \alpha_1 \partial_{\zeta_1} + \alpha_2 \partial_{\zeta_2} + \cdots + \alpha_N \partial_{\zeta_N} = \boldsymbol{\alpha} \cdot \nabla,$$
$$\partial_t = \beta_1 \partial_{\zeta_1} + \beta_2 \partial_{\zeta_2} + \cdots + \beta_N \partial_{\zeta_N} = \boldsymbol{\beta} \cdot \nabla,$$

then we have

$$\partial_x^k \partial_t^l \vartheta(\boldsymbol{\zeta}, \boldsymbol{\tau}) = (\boldsymbol{\alpha} \cdot \nabla)^k (\boldsymbol{\beta} \cdot \nabla)^l \vartheta(\boldsymbol{\zeta}, \boldsymbol{\tau}), \quad k, l = 0, 1, \cdots .$$

10.3. One-periodic waves and asymptotic analysis

Let us first construct one-periodic wave solutions of the NKdV equation (1.1) by using bilinear Báclund transformation (7.6). As a simple case of the theta function (10.1) with $N = 1, s = 0$, we choose F and G as follows

$$\begin{aligned} F &= \vartheta(\zeta, 0, 0|\tau) = \sum_{n \in \mathbb{Z}} \exp(2\pi i n \zeta - \pi n^2 \tau), \\ G &= \vartheta(\zeta, 1/2, 0|\tau) = \sum_{n \in \mathbb{Z}} \exp(2\pi i n (\zeta + 1/2) - \pi n^2 \tau) \\ &= \sum_{n \in \mathbb{Z}} (-1)^n \exp(2\pi i n \zeta - \pi n^2 \tau), \end{aligned} \tag{10.6}$$

where $\zeta = \alpha x + \beta t + \delta$ is the phase variable, and $\tau > 0$ is a positive parameter.

By Theorem 6, the operator $H_1 = D_x^2 - \lambda$ in bilinear equation (7.6) is symmetric, and its corresponding constraint equation in the formula (10.5) automatically vanishes for $\mu = 1$. Meanwhile, $H_2 = D_t D_x^2 - 2w_0 D_x + (4v_0 + 3\lambda) D_t$ are skew-symmetric, and its corresponding constraint equation automatically vanishes for $\mu = 0$. Therefore, the Riemann theta function (10.6) is a solution of the bilinear equation (7.6), provided the following equations

$$\begin{aligned} &\sum_{n \in \mathbb{Z}} \{[4\pi i (n - \mu/2)]^2 \alpha^2 - \lambda\} \exp(-2\pi\tau(n - \mu/2)^2 + \pi i (n - \mu/2))|_{\mu=0} = 0, \\ &\sum_{n \in \mathbb{Z}} \{[4\pi i (n - \mu/2)]^3 \alpha^2 \beta + 8\pi i (n - \mu/2) \alpha w_0 + 4\pi i (n - \mu/2)(4v_0 + 3\lambda)\beta\} \\ &\times \exp(-2\pi\tau(n - \mu/2)^2 + \pi i (n - \mu/2))|_{\mu=1} = 0. \end{aligned} \tag{10.7}$$

hold.

Let

$$\rho = e^{-\pi\tau/2},$$

$$\vartheta_1(\zeta,\rho) = \vartheta(2\zeta, 1/4, -1/2|2\tau) = \sum_{n\in\mathbb{Z}} \rho^{(2n-1)^2} \exp[4i\pi(n-1/2)(\zeta+1/4)],$$

$$\vartheta_2(\zeta,\rho) = \vartheta(2\zeta, 1/4, 0|2\tau) = \sum_{n\in\mathbb{Z}} \rho^{4n^2} \exp[4i\pi n(\zeta+1/4)],$$

then, the equation (10.7) can be written as a linear system about β and λ

$$\begin{aligned} &\vartheta_2''\alpha^2 - \vartheta_2\lambda = 0, \\ &\vartheta_1'''\alpha^2\beta + 2\vartheta_1'\alpha w_0 + (4v_0 + 3\lambda)\vartheta_1'\beta = 0, \end{aligned} \tag{10.8}$$

where the derivative value of $\vartheta_j(\zeta,\rho)$ at $\zeta = 0$ is denoted by simple notations

$$\vartheta_j' = \vartheta_j'(0,\rho) = \frac{d\vartheta_j(\zeta,\rho)}{d\zeta}|_{\zeta=0}, \quad j = 1, 2.$$

It is not hard to see that the system (10.8) admits the following solution for the NKdV equation (1.1)

$$\lambda = \frac{\vartheta_2''\alpha^2}{\vartheta_2}, \quad \beta = \frac{-2\vartheta_1'\vartheta_2 w_0}{\vartheta_1'''\vartheta_2\alpha^2 + 4\vartheta_1'\vartheta_2 v_0 + 3\vartheta_1'\vartheta_2''\alpha^2}. \tag{10.9}$$

So, we obtain the following one-periodic wave solution

$$V = v_0 + 2\partial_x^2 \ln\vartheta(\zeta,0,0|\tau), \quad W = w_0 + 2\partial_x\partial_t \ln\vartheta(\zeta,0,0|\tau), \tag{10.10}$$

where $\zeta = \alpha x + \beta t + \delta$ and parameter β is given by (10.9), while other parameters α, τ, v_0, w_0 are arbitrary. Among the four parameters, the two ones α and τ completely dominate a one-periodic wave.

In summary, one-periodic wave (10.10) is one-dimensional and has two fundamental periods 1 and $i\tau$ in phase variable ζ (see Figure 3).

In the following theorem, we will see that the one-periodic wave solution (10.10) can be broken into soliton solution (6.20) under a long time limit and their relation can be established as follows.

Theorem 13. In the one-periodic wave solution (10.6), the parameter β is given by (10.9), other parameters are chosen as

$$\alpha = \frac{k}{2\pi i}, \quad \delta = \frac{\gamma + \pi\tau}{2\pi i}, \tag{10.11}$$

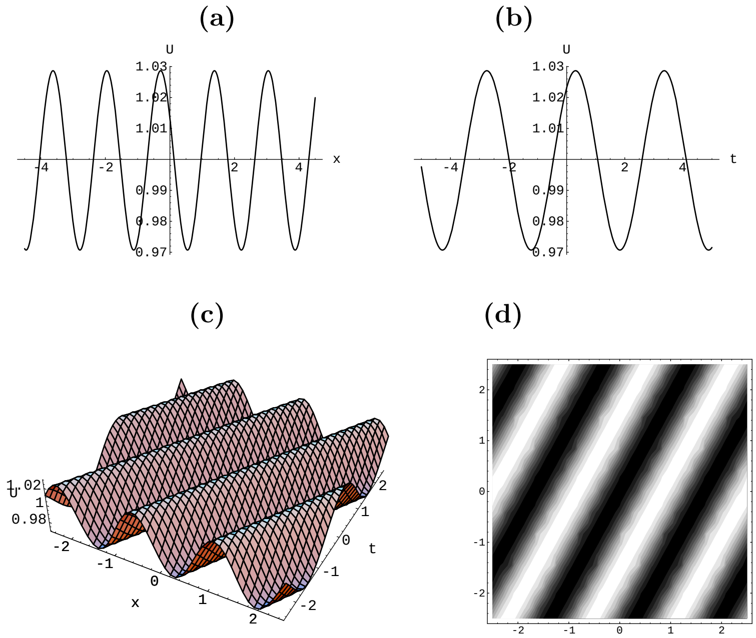

FIGURE 3. One-periodic wave for the NKdV equation (1.1) with parameters: $\alpha = 0.6$, $\tau = 2, v_0 = 0.5, w_0 = 1$. (a) and (b) show that every one-periodic wave is periodic in both x and y directions. (c) Perspective view of the wave. (d) Overhead view of the wave, with contour plot shown. The bright hexagons are crests and the dark hexagons are troughs.

where k_1 and γ are the same as those in (6.20). Then under a small amplitude limit, one-periodic wave solution (10.10) can be broken into the single soliton solutions (6.20), that is,

$$V \longrightarrow v, \quad W \longrightarrow w, \quad \text{as} \quad \rho \to 0. \tag{10.12}$$

In particular, in the case of $v_0 = 0,\ w_0 = 1$, the one-periodic solution (10.5) tends to the kink-type soliton solution (5.2), that is,

$$V \longrightarrow \tilde{v}^I, \quad W \longrightarrow \tilde{w}^I, \quad \text{as} \quad \rho \to 0. \tag{10.13}$$

Proof. Here we use the system (10.8) to analyze asymptotic properties of the one-periodic solution (10.10). Let us expand the coefficients of the system (10.8) as follows

$$\begin{aligned} &\vartheta_1' = -4\pi\rho + 12\pi\rho^9 + \cdots, \quad \vartheta_1''' = 16\pi^3\rho + 432\pi^3\rho^9 + \cdots, \\ &\vartheta_2 = 1 + 2\rho^4 + \cdots, \quad \vartheta_2'' = 32\pi^2\rho^4 + \cdots, \end{aligned} \tag{10.14}$$

Suppose that the solution of the system (10.8) has the following form

$$\begin{aligned} \lambda &= \lambda_0 + \lambda_1\rho + \lambda_2\rho^2 + \cdots = \lambda_0 + o(\rho), \\ \beta &= \beta_0 + \beta_1\rho + \beta_2\rho^2 + \cdots = \beta_0 + o(\rho). \end{aligned} \tag{10.15}$$

Substituting the expansions (10.14) and (10.15) into the system (10.8) and letting $\rho \to 0$, we immediately obtain the following relation

$$\lambda_0 = 0, \quad \beta_0 = \frac{-\alpha w_0}{-2\pi^2\alpha^2 + 2v_0}. \tag{10.16}$$

Combining (10.11) and (10.16) leads to

$$\begin{aligned} &\lambda \longrightarrow 0, \\ &2\pi i\beta \longrightarrow 2\pi i\beta_0 = \frac{-2\pi i\alpha w_0}{-2\pi^2\alpha^2 + 2v_0} = \frac{-2kw_0}{k^2 + 4v_0}, \quad \text{as} \quad \rho \to 0, \end{aligned}$$

or equivalently rewritten as

$$\begin{aligned} \hat{\zeta} &= 2\pi i\zeta - \pi\tau = kx + 2\pi i\beta t + \gamma \\ &\longrightarrow kx - \frac{2kw_0}{k^2 + 4v_0}t + \gamma = \xi, \quad \text{as} \quad \rho \to 0. \end{aligned} \tag{10.17}$$

It remains to verify that the one-periodic wave (10.11) has the same form as the one-soliton solution (6.20) under the limit $\rho \to 0$. Let us expand the function F in the following form

$$F = 1 + \rho^2(e^{2\pi i\zeta} + e^{-2\pi i\zeta}) + \rho^8(e^{4\pi i\zeta} + e^{-4\pi i\zeta}) + \cdots .$$

It follows from (10.11) and (10.17)

$$\begin{aligned} F &= 1 + e^{\hat{\zeta}} + \rho^4(e^{-\hat{\zeta}} + e^{2\hat{\zeta}}) + \rho^{12}(e^{-2\hat{\zeta}} + e^{3\hat{\zeta}}) + \cdots \\ &\longrightarrow 1 + e^{\hat{\zeta}} \longrightarrow 1 + e^{\xi}, \quad \text{as} \quad \rho \to 0. \end{aligned} \tag{10.18}$$

So, combining (10.11) and (10.18) yields

$$\begin{aligned} &v \longrightarrow v_0 + 2\partial_{xx}\ln(1 + e^{\xi}), \\ &w \longrightarrow w_0 + 2\partial_t\partial_x\ln(1 + e^{\xi}), \quad \text{as} \quad \rho \to 0. \end{aligned}$$

Thus, we conclude that the one-periodic solution (10.10) may go to a bell-type soliton solutions (6.20) as the amplitude $\rho \to 0$. □

10.4. Two-periodic waves and asymptotic properties

Let us now consider two-periodic wave solutions to the NKdV equation (1.1). For the case of $N = 2$, $\boldsymbol{s} = \boldsymbol{0}$, $\boldsymbol{\varepsilon} = \boldsymbol{1/2} = (1/2, 1/2)$ in the Riemann theta function (10.1), we choose F and G as follows

$$\begin{aligned} F &= \vartheta(\boldsymbol{\zeta}, \boldsymbol{0}, \boldsymbol{0}|\boldsymbol{\tau}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \exp\{2\pi i\langle \boldsymbol{\zeta}, \boldsymbol{n}\rangle - \pi\langle \boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{n}\rangle\} \\ G &= \vartheta(\boldsymbol{\zeta}, \boldsymbol{1/2}, \boldsymbol{0}|\boldsymbol{\tau}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \exp\{2\pi i\langle \boldsymbol{\zeta} + \boldsymbol{1/2}, \boldsymbol{n}\rangle - \pi\langle \boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{n}\rangle\} \\ &= \sum_{\boldsymbol{n}\in\mathbb{Z}^2} (-1)^{n_1+n_2} \exp\{2\pi i\langle \boldsymbol{\zeta}, \boldsymbol{n}\rangle - \pi\langle \boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{n}\rangle\} \end{aligned} \tag{10.19}$$

where $\boldsymbol{n} = (n_1, n_2) \in Z^2$, $\boldsymbol{\zeta} = (\zeta_1, \zeta_2) \in \mathcal{C}^2$, $\zeta_i = \alpha_j x + \beta_j t + \delta_j$, $j = 1, 2$, and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $\boldsymbol{\beta} = (\beta_1, \beta_2) \in \mathcal{C}^2$. The matrix $\boldsymbol{\tau}$ is a positive definite and real-valued symmetric 2×2 matrix that is,

$$\boldsymbol{\tau} = (\tau_{ij})_{2\times 2}, \quad \tau_{12} = \tau_{21}, \quad \tau_{11} > 0, \quad \tau_{22} > 0, \quad \tau_{11}\tau_{22} - \tau_{12}^2 > 0.$$

According to Theorem 5, constraint equations associated with $H_1 = D_x^2 - \lambda$ and $H_2 = D_t D_x^2 - 2w_0 D_x + (4v_0 + 3\lambda)D_t$ automatically vanish for $(\mu_1, \mu_2) = (0, 1), (1, 0)$ and for $(\mu_1, \mu_2) = (0, 0), (1, 1)$, respectively. Hence, making the theta functions (10.19) satisfy the bilinear equation (7.6) gives the following constraint equations

$$\begin{aligned} &\sum_{n_1,n_2\in\mathbb{Z}} \left[-16\pi^2\langle \boldsymbol{n} - \boldsymbol{\mu}/2, \boldsymbol{\alpha}\rangle^2 - \lambda\right] \exp\{-2\pi\langle \boldsymbol{\tau}(\boldsymbol{n} - \boldsymbol{\mu}/2), \boldsymbol{n} - \boldsymbol{\mu}/2\rangle \\ &+ \pi i \sum_{j=1}^{2} (n_j - \mu_j/2)\}|_{\boldsymbol{\mu}=(\mu_1,\mu_2)} = 0, \quad \text{for} \quad (\mu_1, \mu_2) = (0, 0),\ (1, 1) = 0, \\ &\sum_{n_1,n_2\in\mathbb{Z}} \left[-64\pi^3 i\langle \boldsymbol{n} - \boldsymbol{\mu}/2, \boldsymbol{\alpha}\rangle^2\langle \boldsymbol{n} - \boldsymbol{\mu}/2, \boldsymbol{\beta}\rangle + 8\pi i\langle \boldsymbol{n} - \boldsymbol{\mu}/2, \boldsymbol{\alpha}\rangle w_0 + 4\pi i\langle \boldsymbol{n} - \boldsymbol{\mu}/2, \boldsymbol{\beta}\rangle(4v_0 + 3\lambda)\right] \\ &\times \exp\{-2\pi\langle \boldsymbol{\tau}(\boldsymbol{n} - \boldsymbol{\mu}/2), \boldsymbol{n} - \boldsymbol{\mu}/2\rangle + \pi i \sum_{j=1}^{2} (n_j - \mu_j/2)\}|_{\boldsymbol{\mu}=(\mu_1,\mu_2)} = 0, \\ &\text{for} \quad (\mu_1, \mu_2) = (0, 1),\ (1, 0). \end{aligned} \tag{10.20}$$

Let

$$\rho_{kl} = e^{-\pi\tau_{kl}/2}, k,l = 1,2, \boldsymbol{\rho} = (\rho_{11}, \rho_{12}, \rho_{22})$$

$$\vartheta_j(\boldsymbol{\zeta}, \boldsymbol{\rho}) = \vartheta(2\zeta, \mathbf{1/4}, -\boldsymbol{s}_j/2|2\tau)$$

$$= \sum_{n_1,n_2\in Z} \exp\{4\pi i\langle \boldsymbol{\zeta} + \mathbf{1/4}, \boldsymbol{n} - \boldsymbol{s_j}/2\rangle\} \prod_{k,l=1}^{2} \rho_{kl}^{(2n_k-s_{j,k})(2n_j-s_{j,l})},$$

$$\boldsymbol{s_j} = (s_{j,1}, s_{j,2}), \quad j = 1,2, \quad \boldsymbol{s}_1 = (0,1), \quad \boldsymbol{s_2} = (1,0), \quad \boldsymbol{s_3} = (0,0), \quad \boldsymbol{s_4} = (1,1)$$

then the system (10.20) can be rewritten as a linear system

$$(\boldsymbol{\alpha}\cdot\nabla)^2\vartheta_j - \lambda\vartheta_j = 0, \ j = 3,4, \tag{10.21}$$

$$(\boldsymbol{\beta}\cdot\nabla)(\boldsymbol{\alpha}\cdot\nabla)^2\vartheta_j + 2w_0(\boldsymbol{\alpha}\cdot\nabla)\vartheta_j + (4v_0+3\lambda)(\boldsymbol{\beta}\cdot\nabla)\vartheta_j = 0, \ j = 1,2, \tag{10.22}$$

where ϑ_j represent the derivative values of functions $\vartheta_j(\boldsymbol{\zeta}, \boldsymbol{\rho})$ at $\zeta_1 = \zeta_2 = 0$.

The system (10.22) admits a unique solution

$$\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \left[\frac{\partial(f,g)}{\partial(\zeta_1,\zeta_2)}\right]^{-1} \begin{pmatrix}2w_0(\boldsymbol{\alpha}\cdot\nabla)\vartheta_1\\ 2w_0(\boldsymbol{\alpha}\cdot\nabla)\vartheta_2\end{pmatrix} \tag{10.23}$$

where $\frac{\partial(f,g)}{\partial(\zeta_1,\zeta_2)}$ is the Wronskinan matrix given by

$$\frac{\partial(f,g)}{\partial(\zeta_1,\zeta_2)} = \begin{pmatrix}\partial_{\zeta_1} f & \partial_{\zeta_2} f\\ \partial_{\zeta_1} g & \partial_{\zeta_2} g\end{pmatrix},$$

$$f = [(\boldsymbol{\alpha}\cdot\nabla)^2 + 4v_0 + 3\lambda]\vartheta_1, \quad g = [(\boldsymbol{\alpha}\cdot\nabla)^2 + 4v_0 + 3\lambda]\vartheta_2.$$

With the help of the above (β_1, β_2), we are able to get a two-periodic wave solution to the NKdV equation (1.1)

$$V = v_0 + \partial_x^2 \ln\vartheta(\boldsymbol{\zeta}, \mathbf{0}, \mathbf{0}|\boldsymbol{\tau}), \quad W = w_0 + \partial_x\partial_t\vartheta(\boldsymbol{\zeta}, \mathbf{0}, \mathbf{0}|\boldsymbol{\tau}), \tag{10.24}$$

where $\alpha_1, \alpha_2, \tau_{12}, \delta_1$ and δ_2 are arbitrary parameters, while other parameters β_1, β_2 and τ_{11}, τ_{22} are given by (10.23) and (10.21), respectively.

In summary, the two-periodic wave (10.24) is a direct generalization of two one-periodic waves. Its surface pattern is two-dimensional with two phase variables ζ_1 and ζ_2. The two-periodic wave (10.24) has 4 fundamental periods $\{e_1, e_2\}$ and $\{i\tau_1, i\tau_2\}$ in (ζ_1, ζ_2), and is spatially periodic in two directions ζ_1, ζ_2. Its real part is not periodic in θ_1 direction, while its imaginary part and modulus are all periodic in both x and t directions.

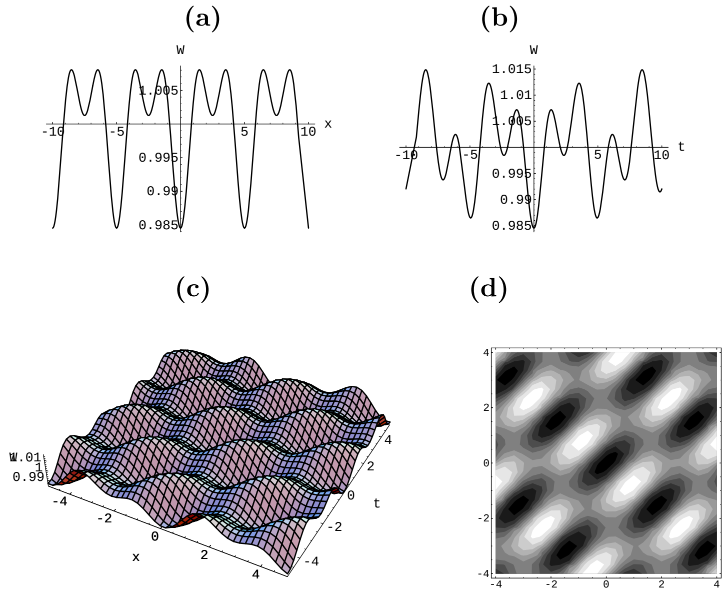

FIGURE 4. Two-periodic wave for the NKdV equation (1.1). (a) and (b) show that every one-periodic wave is periodic in both x- and y-directions. (c) Perspective view of the wave. (d) Overhead view of the wave, with contour plot shown. The bright hexagons are crests and the dark hexagons are troughs.

Finally, we study the asymptotic properties of the two-periodic solution (10.24). In a similar way to Theorem 5, we figure out the relation between the two-periodic solution (10.24) and the two-soliton solution (6.21) as follows.

Theorem 14. Assume that (β_1, β_2) is a solution of the system (10.22), and in the two-periodic wave solution (10.24), parameters $\alpha_j, \delta_j, \tau_{12}$ are chosen as

$$\alpha_j = \frac{k_j}{2\pi i}, \quad \delta_j = \frac{\gamma_j + \pi\tau_{jj}}{2\pi i}, \quad \tau_{12} = -\frac{A_{12}}{2\pi}, \quad j = 1, 2, \tag{10.25}$$

where $k_j, \gamma_j, j = 1, 2$ and A_{12} are those given in (6.21). Then, we have the following asymptotic relations

$$\begin{aligned} &\lambda \longrightarrow 0, \quad \zeta_j \longrightarrow \frac{\eta_j + \pi\tau_{jj}}{2\pi i}, \quad j = 1, 2, \\ &F \longrightarrow 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1 + \eta_2 + A_{12}}, \quad \text{as} \quad \rho_{11}, \rho_{22} \to 0. \end{aligned} \tag{10.26}$$

So, the two-periodic wave solution (10.24) just tends to the two-soliton solution (6.21) under a limit condition

$$V \longrightarrow v, \quad W \longrightarrow w, \quad \text{as} \quad \rho_{11}, \rho_{22} \to 0.$$

Proof. Using (10.20), we may expand the function F in the following explicit form

$$F = 1 + (e^{2\pi i\zeta_1} + e^{-2\pi i\zeta_1})e^{-\pi\tau_{11}} + (e^{2\pi i\zeta_2} + e^{-2\pi i\zeta_2})e^{-\pi\tau_{22}}$$
$$+ (e^{2\pi i(\zeta_1+\zeta_2)} + e^{-2\pi i(\zeta_1+\zeta_2)})e^{-\pi(\tau_{11}+2\tau_{12}+\tau_{22})} + \cdots$$

Furthermore, adopting (10.25) and making a transformation we infer that

$$F = 1 + e^{\hat{\zeta}_1} + e^{\hat{\zeta}_2} + e^{\hat{\zeta}_1+\hat{\zeta}_2-2\pi\tau_{12}} + \rho_{11}^4 e^{-\hat{\zeta}_1} + \rho_{22}^4 e^{-\hat{\zeta}_2} + \rho_{11}^4\rho_{22}^4 e^{-\hat{\zeta}_1-\hat{\zeta}_2-2\pi\tau_{12}} + \cdots$$
$$\longrightarrow 1 + e^{\hat{\zeta}_1} + e^{\hat{\zeta}_2} + e^{\hat{\zeta}_1+\hat{\zeta}_2+A_{12}}, \quad \text{as} \quad \rho_{11}, \rho_{22} \to 0,$$

where $\hat{\zeta}_j = \alpha_j x + \hat{\beta}_j t + \delta_j, \quad j = 1, 2$, and $\hat{\beta}_j = 2\pi i\beta_j, j = 1, 2$.

Now, we need to prove

$$\hat{\beta}_j \longrightarrow \frac{-2k_j w_0}{k_j^2 + 4v_0}, \quad \hat{\zeta}_j \longrightarrow \xi_j, \quad j = 1, 2, \quad \text{as} \quad \rho_{11}, \rho_{22} \to 0. \tag{10.27}$$

As in the case of $N = 1$, the solution of the system (10.23) has the following form

$$\begin{aligned}
\beta_1 &= \beta_{1,0} + \beta_{1,1}\rho_{11} + \beta_{2,2}\rho_{22} + o(\rho_{11}, \rho_{22}),\\
\beta_2 &= \beta_{2,0} + \beta_{2,1}\rho_{11} + \beta_{2,2}\rho_{22} + o(\rho_{11}, \rho_{22}),\\
\lambda &= \lambda_0 + \lambda_1\rho_{11} + \lambda_2\rho_{22} + o(\rho_{11}, \rho_{22}).
\end{aligned} \tag{10.28}$$

Expanding functions $\vartheta_j, \ j = 1, 2, 3, 4$ in equations (10.21) and (10.22) with substitution of assumption (10.28), and letting $\rho_{11}, \rho_{22} \longrightarrow 0$, we will obtain

$$\begin{aligned}
&\lambda_0 = 0,\\
&16\pi i(-\pi^2\alpha_1^2 + v_0)\beta_{1,0} - 8\pi i w_0\alpha_1 = 0,\\
&16\pi i(-\pi^2\alpha_2^2 + v_0)\beta_{2,0} - 8\pi i w_0\alpha_2 = 0.
\end{aligned} \tag{10.29}$$

Using (10.28) and (10.29), we conclude that

$$\begin{aligned}
&\lambda = o(\rho_{11}, \rho_{22}) \longrightarrow 0,\\
&\beta_j = \frac{-2k_j w_0}{k_j^2 + 4v_0} + o(\rho_{11}, \rho_{22}) \longrightarrow \frac{-2k_j w_0}{k_j^2 + 4v_0}, \quad \text{as} \quad \rho_{11}, \rho_{22} \to 0,
\end{aligned}$$

and therefore we have (10.26). So, the two-periodic wave solution (10.24) tends to the two-soliton solution (6.21) as $\rho_{11}, \rho_{22} \to 0$. □

10.5. Multi-periodic wave solutions

The system (10.5) indicates that constructing multi-periodic wave solutions depends on the solvability of the system (10.5). Obviously, the number of constraint equations of the type (10.5) is $2^{N-1}+1$. On the other hand, we have $\frac{1}{2}N(N+1)+3N+3$ parameters $\tau_{ii}, \tau_{ij}, \alpha_i, \omega_i, \lambda, u_0, v_0$. Among them, $2N$ parameters τ_{ii}, α_i may be the given parameters relate to the amplitudes and wave numbers of N-periodic waves. Therefore, the number of the unknown parameters is $\frac{1}{2}N(N+1)+N+3$ while $\frac{1}{2}N(N+1)$ parameters τ_{ij}, implicitly appearing in the series form, can not to be solved explicitly in general. So, the number of the explicit unknown parameters is only $N+3$, and the number of equations is larger than the unknown parameters in the case of $N>4$. This fact means that if equation (10.5) is satisfied, then we have at least N-periodic wave solutions $(N \leq 4)$. In this paper, we only consider one- and two-periodic wave solutions of the NKdV equation (1.1). There are still certain computation difficulties in the calculation for the case of $N>2$, which will be studied in the future.

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